Parameterized Complexity of Computing Maximum Minimal Blocking and Hitting Sets

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Abstract

A *blocking set* in a graph G is a subset of vertices that intersects every maximum independent set of G. Let $\mathsf{mmbs}(G)$ be the size of a maximum (inclusion-wise) minimal blocking set of G. This parameter has recently played an important role in the kernelization of Vertex Cover with structural parameterizations. We provide a panorama of the complexity of computing mmbs parameterized by the natural parameter and the independence number of the input graph. We also consider the closely related parameter mmhs, which is the size of a maximum minimal hitting set of a hypergraph. Finally, we consider the problem of computing mmbs parameterized by treewidth, especially relevant in the context of kernelization. Since a blocking set intersects every *maximum-sized* independent set of a given graph and properties involving counting the sizes of arbitrarily large sets are typically non-expressible in monadic second-order logic, its tractability does not seem to follow from Courcelle's theorem. Our main technical contribution is a fixed-parameter tractable algorithm for this problem.

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1 Introduction

Given a graph G, we denote by $\alpha(G)$ the maximum size of an *independent set* of G, that is, of a set of pairwise non-adjacent vertices. For the sake of conciseness, we abbreviate "independent set" as is, and "maximum independent set" as mis. A set $B \subseteq V(G)$ is a *blocking set*, abbreviated as bs, of G if $\alpha(G \setminus B) < \alpha(G)$, where $G \setminus B = G[V(G) \setminus B]$. Equivalently, B is a blocking set of G if for every mis $I^* \subseteq V(G)$, $I^* \cap B \neq \emptyset$. In this work we are interested in (inclusion-wise) minimal blocking sets, which we abbreviate as mbs. We denote by $\mathsf{mmbs}(G)$ the maximum size of an mbs of G, and by Maximum Minimal Blocking Set (MMBS for short) the problem where, given a graph G and an integer β, the objective is to decide whether $\mathsf{mmbs}(G) \geq \beta$. The main objective of this paper is to study the parameterized complexity of MMBS. As discussed below, this problem is strongly related to the Maximum Minimal Hitting Set (MMHS) problem, for which we also present several results.

Role of maximum minimal blocking sets in kernelization. Given a graph G, a set of vertices $S \subseteq V(G)$ is a *vertex cover* if it contains at least one endpoint of every edge. The Vertex Cover (VC for short) problem asks, given a graph G and an integer k, if there is a vertex cover S of G such that $|S| \leq k$. For a fixed graph class $\mathcal{F}$, the VC/dist-to-$\mathcal{F}$ parameterized problem is defined as follows. The input is a triple (G, X, k) where G is a graph, $X \subseteq V(G)$, and $G \setminus X$ belongs to $\mathcal{F}$. The set X is often referred to as a *modulator* to $\mathcal{F}$, and $|X|$ as the *distance* of G to $\mathcal{F}$. The objective of the problem is to decide whether G admits a vertex cover of size at most k, and the parameter is $|X|$. A *kernel* of vertex size f for this problem is a polynomial-time algorithm that, given an input (G, X, k), outputs an equivalent instance (G', X', k') with $|V(G')| \leq f(|X|)$. Informally, such a kernel compresses the input graph G to a smaller graph G' whose size is bounded by a function f depending only on $|X|$. If f is a polynomial (resp. linear) function, we speak of a *polynomial* (resp. *linear*) kernel. We refer the reader to Section 2 for formal definitions. The VC/dist-to-$\mathcal{F}$ problem has been defined by Jansen and Bodlaender [23] for $\mathcal{F}$ being the class of forests as a way to improve the linear kernel for Vertex Cover parameterized by the standard parameter k. Their main result is a polynomial kernel for VC/dist-to-$\mathcal{F}$ (for $\mathcal{F}$ being the forests).

This result triggered a long line of follow-up research, which aimed to find the most general graph families $\mathcal{F}$ such that VC/dist-to-$\mathcal{F}$ admits a polynomial kernelization [19]. Several results were proved for specific families $\mathcal{F}$ such as those of degree at most two, of bounded treedepth, pseudo-forests (see [9, 19] for a complete list of references), and a major open question in this area is to find a characterization of the families $\mathcal{F}$ for which VC/dist-to-$\mathcal{F}$ admits a polynomial kernel [9]. This is where parameter mmbs comes into play, as we proceed to explain.

Kernelization algorithms for VC/dist-to-$\mathcal{F}$ usually proceed in two steps. In step 1, they reduce the number of connected components of $G \setminus X$ to a polynomial in $|X|$, and in step 2 they reduce the size of each connected component of $G \setminus X$ to a polynomial in $|X|$ as well. Minimal blocking sets have been introduced in the seminal paper of Jansen and Bodlaender [23] for the case of $\mathcal{F}$ being the class of forests as a handy tool to achieve step 1. After that, this notion has been generalized and reused for example in [9, 10, 21], finally leading to the following black box tool for step 1, where $\mathsf{mmbs}(\mathcal{F}) = \sup_{G \in \mathcal{F}} \mathsf{mmbs}(G)$.

▶ **Theorem 1** (Hols et al. [21]). *Let $\mathcal{F}$ be a hereditary graph class on which* VC *can be solved in polynomial time. There is a polynomial-time algorithm that, given an instance (G, X, k) of* VC/dist-to-$\mathcal{F}$*, returns an equivalent instance (G_0, X, k_0) of* VC/dist-to-$\mathcal{F}$ *such that $G_0 \setminus X \in \mathcal{F}$ and has $\mathcal{O}(|X|^{\mathsf{mmbs}(\mathcal{F})})$ connected components.*

Informally, Theorem 1 states that, if $\mathsf{mmbs}(\mathcal{F})$ is bounded by a constant, then "half" of the kernelization algorithm can be done automatically. Moreover, it has been shown that $\mathsf{mmbs}(\mathcal{F})$ being bounded by a constant is necessary in order to obtain a polynomial kernel:

▶ **Theorem 2** (Hols et al. [21]). *Unless* NP $\subseteq$ coNP/poly*,* VC/dist-to-$\mathcal{F}$ *does not admit a kernel of size $\mathcal{O}(|X|^{\mathsf{mmbs}(\mathcal{F})-\varepsilon})$ for any $\varepsilon > 0$.*

These two theorems suggest that mmbs might be the right candidate to characterize graph classes $\mathcal{F}$ for which VC/dist-to-$\mathcal{F}$ admits a polynomial kernel. However, it turns out that there exists a class $\mathcal{F}$ where $\mathsf{mmbs}(\mathcal{F})$ is bounded by a constant, but for which there is no polynomial kernel for VC/DIST-TO-$\mathcal{F}$ under standard complexity assumptions [21]. Nevertheless, for minor-closed families[1], the following theorem shows that mmbs is indeed the correct parameter in order to characterize the existence of polynomial kernels for VC/DIST-TO-$\mathcal{F}$.

▶ **Theorem 3** (Bougeret et al. [9]). *If $\mathcal{F}$ is a minor-closed graph class, then* VC/DIST-TO-$\mathcal{F}$ *admits a polynomial kernel if and only if* $\mathsf{mmbs}(\mathcal{F})$ *is bounded by a constant.*

To summarize this discussion, for general graph classes $\mathcal{F}$, having bounded $\mathsf{mmbs}(\mathcal{F})$ is necessary but not sufficient, although having bounded $\mathsf{mmbs}(\mathcal{F})$ yields "half" of the kernel. For minor-closed classes $\mathcal{F}$, having bounded $\mathsf{mmbs}(\mathcal{F})$ is indeed the correct characterization. These results explain the recent interest in computing $\mathsf{mmbs}(\mathcal{F})$ for different classes $\mathcal{F}$ [21], and thus our motivation to study the complexity of the MMBS problem.

Let us also mention that computing mmbs can in addition be useful when implementing any of the previously mentioned kernels. Indeed, given an instance (G, X, k) of VC/DIST-TO-$\mathcal{F}$ with $|V(G)| = n$, the algorithm behind Theorem 1 takes as additional input the value $\mathsf{mmbs}(\mathcal{F})$ and outputs the claimed equivalent instance in time $n^{\mathsf{mmbs}(\mathcal{F})+\mathcal{O}(1)}$. However, when implementing this algorithm, we can rather first compute $\mathsf{mmbs}(G \setminus X)$, and use the algorithm of Theorem 1 with additional input $\mathsf{mmbs}(G \setminus X)$ instead of $\mathsf{mmbs}(\mathcal{F})$ (note that $\mathsf{mmbs}(G \setminus X) \leq \mathsf{mmbs}(\mathcal{F})$, potentially much smaller), and thus obtain a running time $n^{\mathsf{mmbs}(G\setminus X)+\mathcal{O}(1)}$, and an equivalent instance (G_0, X, k_0), where $G_0 \setminus X \in \mathcal{F}$, with $\mathcal{O}(|X|^{\mathsf{mmbs}(G\setminus X)})$ connected components.

Contribution and related work. In what follows we present our contribution and relate it to previous work, by considering each parameterization of the studied problems separately.

Choice of the parameters. As the VC/DIST-TO-$\mathcal{F}$ problem has only been considered for graph classes $\mathcal{F}$ where MAXIMUM INDEPENDENT SET (IS for short) can be solved in polynomial time, we also incorporate this assumption in this work, hence motivating the parameterization of MMBS by combinations of α (i.e., the size of a mis of the input graph) and the threshold β (the solution size). Moreover, since in all the previously mentioned cases where VC/DIST-TO-$\mathcal{F}$ has a polynomial kernel [9, 19] the graphs in the class $\mathcal{F}$ have bounded *treewidth*, we also consider the MMBS problem parameterized by the treewidth of the input graph G.

Problems related to computing mmbs. We denote by MAXIMUM MINIMAL HITTING SET (MMHS for short) the problem where, given a hypergraph $\mathcal{H}$ and an integer β, the objective is to decide whether $\mathsf{mmhs}(\mathcal{H}) \geq \beta$, where $\mathsf{mmhs}(\mathcal{H})$ is the size of a largest minimal hitting set of $\mathcal{H}$, that is, an (inclusion-wise) minimal set of vertices of $\mathcal{H}$ containing at least one vertex of every hyperedge. A *dominating set* in a graph G is a subset of vertices $S \subseteq V(G)$ such that every vertex in $V(G) \setminus S$ has a neighbor in S. We denote by UPPER DOMINATING SET (UP-DOM for short) the problem of computing a maximum (inclusion-wise) minimal dominating set in an input graph G. As pointed out by Bazgan et al. [2], UP-DOM is a special case of MMHS, as we can create a hyperedge for each closed neighborhood of the vertices in G, implying that the negative results stated below for UP-DOM transfer directly to MMHS. As mentioned before, we only consider graph classes $\mathcal{F}$ where IS can be solved in polynomial time. A natural special case of such classes is when α is constant, and in this case MMBS reduces to MMHS by simply generating in time $n^{\alpha+\mathcal{O}(1)}$ a hyperedge for each mis of G. Let us now define parameterizations for MMBS and MMHS. A parameterization is, in a

[1] A graph class is *minor-closed* if any minor of a member in the class also belongs to it, and a graph H is a *minor* of a graph G is H can be obtained from a subgraph of G by contracting edges.

nutshell, a function mapping instances of a decision problem to non-negative integers (see Section 2 for the details). For MMBS, recall that we have a graph G and a positive integer k as input and we wish to decide whether $\mathsf{mmbs}(G) \geq k$. Let $\mathcal{G}$ denote the set of all graphs. We define, with slight abuse of notation, the parameters $\alpha : \mathcal{G} \times \mathbb{N} \to \mathbb{N}$ as $\alpha(G, k) = \alpha(G)$ and $\beta : \mathcal{G} \times \mathbb{N} \to \mathbb{N}$ as $\beta(G, k) = \mathsf{mmbs}(G)$. Similarly, for MMHS, we define the parameters α as $\alpha(\mathcal{H}, k) = \max_{H \in E(\mathcal{H})} |H|$ and β as $\beta(\mathcal{H}, k) = \mathsf{mmhs}(\mathcal{H})$.

The first objective of this paper is to obtain a complete landscape of the parameterized complexity of MMBS and MMHS under different combinations of α and β to compare the behavior of these two problems. To the best of our knowledge, the parameterized complexity of MMBS has not been considered before in the literature. On the other hand, the MMHS problem has received considerable attention, especially concerning the enumeration of minimal hitting sets, as it allows to construct the so-called *dual* hypergraph, that is, the hypergraph having a hyperedge for every minimal hitting set of the original hypergraph. Let us now present our contributions together with the related work about the parameterized complexity of MMHS, for each different parameterization that we consider.

Parameterization by α or β separately. When parameterizing by α only, both MMBS and MMHS are para-NP-hard, meaning NP-hard for fixed values of the parameter. Indeed, the particular case $\alpha = 2$ of MMHS corresponds to the Maximum Minimal Vertex Cover (MMVC for short) problem, which is NP-hard [8]. When parameterizing by β only, we show in Proposition 6 that MMBS is para-NP-hard, whereas MMHS is W[1]-hard [2] and XP [5]. As discussed in Section 3, the W[1]-hardness proof of [2] implies that, unless the Exponential Time Hypothesis (ETH for short) fails, Up-Dom cannot be solved in time $f(k) \cdot n^{o(\sqrt{k})}$ on n-vertex graphs for any computable function f. We improve this lower bound by showing in Theorem 7 that Up-Dom cannot be solved in time $f(k) \cdot n^{o(k)}$ for any computable function f, implying the same result (replacing k by β) for MMHS. We point out that very recently and independently from our work, this improved lower bound for Up-Dom has also been proved by Dublois et al. [18].

Parameterization by one parameter while fixing the other. When fixing α and parameterizing by β, MMBS reduces to MMHS, which was known to be FPT [15], and for which we provide in Proposition 16 a polynomial kernel with $\mathcal{O}(\beta^\alpha)$ vertices, generalizing the known quadratic kernel for MMVC [8, 20]. When fixing β and parameterizing by α, we show in Proposition 6 that MMBS is W[1]-hard, whereas MMHS is FPT (as it is even FPT parameterized by the sum as explained in the next paragraph).

Parameterization by the sum. Finally, when parameterizing by $\alpha + \beta$, the hardness result given in Proposition 6 (i.e., parameterizing by α for fixed β) implies that MMBS is W[1]-hard, whereas MMHS is FPT for the following reasons. We first provide in Corollary 18 a simple FPT algorithm for MMHS that reduces to an extension problem considered by Bläsius et al. [5], and then design in Theorem 24 a more involved ad-hoc algorithm to improve the running time to $\mathcal{O}^*(2^{\alpha\beta})$, where the $\mathcal{O}^*$-notation hides multiplicative polynomial terms (see Section 2).

Our results considering parameters α and β are summarized in Table 1.

Parameterization by treewidth. Let us now turn to the second objective of this paper, which is the parameterization by treewidth. It is known that both Up-Dom [2] and MMVC [8] are FPT parameterized by treewidth, but none of these results implies the same result for MMBS. We also mention that that the problem of finding a maximum minimal set intersecting all maximum cliques of a graph is FPT parameterized by treewidth [25], implying that MMBS

Parameter	MMBS	MMHS
β	para-NP-hard (Proposition 6)	XP ([5], Corollary 18) W[1]-hard [2] $\nexists\ f(\beta) \cdot (\lvert V(\mathcal{H})\rvert + \lvert E(\mathcal{H})\rvert)^{o(\beta)}$ (Corollary 11)
α	para-NP-hard (Proposition 6)	para-NP-hard (MMVC for $\alpha = 2$) [8]
β (α fixed)	Reducible to MMHS in time $n^{\alpha+\mathcal{O}(1)}$	Kernel with $\mathcal{O}(\beta^\alpha)$ vertices (Proposition 16) FPT in $\mathcal{O}^*(\alpha^\beta)$ [15] ($\mathcal{O}^*$ hides $n^{f(\alpha)}$)
α (β fixed)	W[1]-hard (Proposition 6) XP (reduction to MMHS)	FPT (as it is FPT by $\alpha + \beta$)
$\alpha + \beta$	W[1]-hard (Proposition 6) XP (reduction to MMHS)	FPT (Corollary 18 and Theorem 24)

Table 1 Parameterized complexity of MMBS and MMHS with parameters α and β. The main differences between MMBS and MMHS show up in the parameterizations by β and $\alpha + \beta$.

is FPT parameterized by treewidth of the complement of the input graph. We prove in Theorem 51 that MMBS is FPT parameterized by treewidth, which is the main technical result of this paper. Let us mention that MMBS does not seem to be (at least, easily) expressible in monadic second-order logic, due to the fact that the a blocking set in a graph G is defined so that it intersects every *maximum-sized* independent set of G, and properties involving counting the *sizes* of arbitrarily large sets are typically non-expressible in monadic second-order logic [7, 13]. This is why, in order to deduce that MMBS is FPT parameterized by treewidth, we cannot directly apply Courcelle's theorem [12], and we need to design an ad-hoc algorithm that is quite involved, needing a number of technical lemmas. In Section 4.1 we discuss the list of inputs of our dynamic programming algorithm, along with several examples to illustrate the difficulties that motivate their choice.

Organization. In Section 2 we provide preliminaries about graphs, parameterized complexity and treewidth, the formal statement of the considered problems, and we state in Lemma 4 several useful properties of minimal blocking and hitting sets. Section 3 is devoted to parameterizations by α and β, and Section 4 to the algorithm for MMBS parameterized by treewidth. We conclude the article in Section 5 with some directions for further research.

2 Preliminaries

Graphs and functions. We only provide here basic definitions and refer the reader to [16] for any missing definitions about graphs. We only consider finite simple graphs with no loops nor multiple edges. For a graph G and a vertex $v \in V(G)$, we denote by $N_G(v)$ the set of vertices of G adjacent to v and, for a subset $S \subseteq V(G)$, we let $N_S(v) = N_G(v) \cap S$. When the graph G is clear from the context, we may omit the subscript. Given $B \subseteq V(G)$, we denote $G \setminus B = G[V(G) \setminus B]$ where $G[X]$ denotes the graph induced by $X \subseteq V(G)$. We denote a *triangle*, that is, a complete graph on three vertices, on vertices u, v, w by (u, v, w). Given a graph G, we say that $X \subseteq V(G)$ is a *vertex cover* if for any edge $e \in E(G)$, $e \cap X \neq \emptyset$, and that X is a *dominating set* if for every $v \in V(G) \setminus X$, there exists $u \in X$ such that $\{u, v\} \in E$. Given a hypergraph $\mathcal{H}$, we say that $I \subseteq V(\mathcal{H})$ is an *independent set* if for every $H \in E(\mathcal{H})$, $H \nsubseteq I$, and that $X \subseteq V(\mathcal{H})$ is a *hitting set* if for every $S \in E(\mathcal{H})$, $S \cap X \neq \emptyset$. A graph class is *hereditary* if it is closed under induced subgraphs.

If a set A is partitioned into pairwise disjoint subsets $A_1, \ldots, A_k$, we denote it by $A = A_1 \uplus \cdots \uplus A_k$. If A is a set, we denote by 2^A the collection containing all the subsets of A. Given a function $f : A \to B$ and a subset $A' \subseteq A$, we denote by $f_{|A'}$ the restriction of f to A'. For a positive integer k, we let $[k]$ be the set containing every integer i such that $1 \leq i \leq k$.

For a function f mapping graphs to integers (such as the parameterizations α and β discussed before) and a class of graphs $\mathcal{F}$, we define $f(\mathcal{F}) = \sup_{G \in \mathcal{F}} f(G)$. Given two functions f_1 and f_2, mapping instances I of a problem to $\mathbb{N}$), if there exists a polynomial p such that for every instance I, $f_1(I) \leq f_2(I) \cdot p(|I|)$, where $|I|$ is the size of I, then we write $f_1 = \mathcal{O}^*(f_2)$.

Parameterized complexity. We refer the reader to [14, 17] for basic background on parameterized complexity, and we recall here only some basic definitions. A *parameterized problem* is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is some fixed alphabet. For an instance $I = (x, k) \in \Sigma^* \times \mathbb{N}$, k is called the *parameter*. Given a classical (non-parameterized) decision problem $L_c \subseteq \Sigma^*$ and a function $\kappa : \Sigma^* \to \mathbb{N}$, we denote by $L_c/\kappa = \{(x, \kappa(x)) \mid x \in L_c\}$ the associated parameterized problem.

A parameterized problem L is *fixed-parameter tractable* (FPT) if there exists an algorithm $\mathcal{A}$, a computable function f, and a constant c such that given an instance $I = (x, k)$, $\mathcal{A}$ (called an FPT *algorithm*) correctly decides whether $I \in L$ in time bounded by $f(k) \cdot |I|^c$. For instance, the Vertex Cover problem parameterized by the size of the solution is FPT.

A parameterized problem L is XP if there exists an algorithm $\mathcal{A}$ (called an *XP algorithm*) and two computable functions f and g such that given an instance $I = (x, k)$, $\mathcal{A}$ (called an XP *algorithm*) correctly decides whether $I \in L$ in time bounded by $f(k) \cdot |I|^{g(k)}$. For instance, the Independent Set problem parameterized by the size of the solution is XP.

Within parameterized problems, the W-hierarchy may be seen as the parameterized equivalent to the class NP of classical decision problems. Without entering into details (see [14, 17] for the formal definitions), a parameterized problem being W[1]-*hard* can be seen as a strong evidence that this problem is *not* FPT. The canonical example of W[1]-hard problem is Independent Set parameterized by the size of the solution.

The most common way to transfer W[1]-hardness is via parameterized reductions. A *parameterized reduction* from a parameterized problem L_1 to a parameterized problem L_2 is an algorithm that, given an instance (x, k) of L_1, outputs an instance (x', k') of L_2 such that

- (x, k) is a yes-instance of L_1 if and only if (x', k') is a yes-instance of L_2,
- $k' \leq g(k)$ for some computable function g, and
- the running time is bounded by $f(k) \cdot |x|^{\mathcal{O}(1)}$ for some computable function f

If L_1 is W[1]-hard and there is a parameterized reduction from L_1 to L_2, then L_2 is W[1]-hard as well.

A parameterized problem is *para*-NP-*hard* if it is NP-hard for some fixed value of the parameter, implying in particular that the problem cannot be in XP unless P = NP.

A *kernelization algorithm* for a parameterized problem L is an algorithm $\mathcal{A}$ that, given an instance (x, k) of L, generates in polynomial time an equivalent instance (x', k') of Q such that $|x'| + k' \leq f(k)$, for some computable function $f : \mathbb{N} \to \mathbb{N}$. If $f(k)$ is bounded from above by a polynomial function, we say that L admits a *polynomial* kernel. In particular, if $f(k)$ is bounded by a linear (resp. quadratic) function, then we say that L admits a *linear* (resp. *quadratic*) kernel.

The Exponential Time Hypothesis (ETH for short) of Impagliazzo et al. [22] is a complexity assumption implying that the 3-SAT problem cannot be solved in time $2^{o(n)}$ restricted to formulas with n variables.

List of considered problems. We denote by IS the Maximum Independent Set problem where, given a graph G and an integer k, the objective is to decide whether $\alpha(G) \geq k$, and by

CIS the Multicolored Independent Set problem, where given graph G and an integer k such that $V(G)$ is partitioned into k cliques $\{V_i \mid i \in [k]\}$, the goal is to decide whether $\alpha(G) \geq k$.

In the two following problems, recall that $\mathsf{mmbs}(G)$ (resp. $\mathsf{mmhs}(\mathcal{H})$) denotes the size of a largest minimal blocking set of G (resp. largest minimal hitting set of $\mathcal{H}$).

Maximum Minimal Blocking Set (MMBS)
Input: A graph G and a positive integer β.
Question: $\mathsf{mmbs}(G) \geq \beta$?

Maximum Minimal Hitting Set (MMHS)
Input: A hypergraph $\mathcal{H}$ and a positive integer β.
Question: $\mathsf{mmhs}(\mathcal{H}) \geq \beta$?

▶ **Property 1.** *If $(\mathcal{H}, \beta)$ is an instance of* MMHS*, then there is an equivalent instance $(\mathcal{H}', \beta)$ satisfying that there is no pair of hyperedges $H_1, H_2 \in E(\mathcal{H}')$ such that $H_1 \subsetneq H_2$. Moreover, such an equivalent instance can be constructed in polynomial time.*

Proof: Let $(\mathcal{H}, \beta)$ be an instance of MMHS such that $H_1, H_2 \in E(\mathcal{H})$ and $H_1 \subsetneq H_2$. Let $\mathcal{H}'$ be obtained from $\mathcal{H}$ by removing H_2. We claim that S is a minimal hitting set of $\mathcal{H}$ if, and only if, S is a minimal hitting set of $\mathcal{H}'$.

Let S be a minimal hitting set of $\mathcal{H}$. Since $E(\mathcal{H}') = E(\mathcal{H}) \setminus \{H_2\}$, it follows that S is indeed a hitting set of $\mathcal{H}'$. To prove that S is a minimal hitting set of $\mathcal{H}'$, by contradiction let $s \in S$ such that $S_- = S \setminus \{v\}$ is a hitting set of $\mathcal{H}'$. In particular, there is a vertex $h_1 \in H_1 \cap S_- \subsetneq H_2$. Thus, S_- would be a hitting set of $\mathcal{H}$ since $H_1 \subsetneq H_2$ and $E(\mathcal{H}') = E(\mathcal{H}) \setminus \{H_2\}$, contradicting the minimality of S.

Conversely, let S' be a minimal hitting set of $\mathcal{H}'$. As before, there is $h_1 \in H_1 \cap S' \subsetneq H_2$. Thus, S' is a hitting set of $\mathcal{H}$. To prove that S' is a minimal hitting set of $\mathcal{H}$, assume that $S'_- = S' \setminus \{v\}$ is a hitting set of $\mathcal{H}$, for some $v \in S'$. Since $E(\mathcal{H}') = E(\mathcal{H}) \setminus \{H_2\}$, we have that $S'_- = S' \setminus \{v\}$ would be a hitting set of $\mathcal{H}'$, contradicting the choice of S'. ◻

Thus, by Property 1 we can always assume that no hyperedge is included in another. Moreover, following the definition of hypergraph given in [4], we also assume that no hyperedge is empty and that each vertex belongs to some hyperedge. We formalize these assumptions in the following observation for further reference.

▶ **Observation 1.** *Whenever $(\mathcal{H}, \beta)$ is an instance of* MMHS *in this work, we assume that:*
1. *For every $H \in E(\mathcal{H})$, $H \neq \emptyset$;*
2. *For every $v \in V(\mathcal{H})$, there is $H \in E(\mathcal{H})$ such that $v \in H$;*
3. *For every $H_1, H_2 \in E(\mathcal{H})$, it is not true that $H_1 \subsetneq H_2$.*

While the third claim is a consequence of Property 1, note that the two first claims of Observation 1 could also be assumed without loss of generality. Indeed, if some hyperedge is empty, then no hitting set exists; and if some vertex does not belong to any hyperedge, then it does not belong to any minimal hitting set and could be safely removed.

The problem Maximum Minimal Vertex Cover (MMVC) corresponds to the restriction of MMHS to instances where all hyperedges have size two, i.e., graphs. For a fixed positive integer α, we also define α-MMHS as the MMHS problem restricted to instances whose hypergraph $\mathcal{H}$ is such that $|H| \leq \alpha$ for every $H \in E(\mathcal{H})$, and α-MMBS as the MMBS problem restricted to instances whose graph G is such that $\alpha(G) \leq \alpha$. Notice that in any FPT or kernel algorithm for α-MMHS or for α-MMBS, as α is fixed, the running time given using the $\mathcal{O}^*$-notation might typically hide a term $n^{f(\alpha)}$, where n is the number of vertices of the graph or hypergraph under consideration. Finally, we define MMBS$^=$ (resp. MMBS$^\leq$) as the MMBS problem where the objective is to decide whether $\mathsf{mmbs}(G) = \beta$ (resp. $\mathsf{mmbs}(G) \leq \beta$).

Extension-MMHS (Ext-MMHS)
Input: A hypergraph $\mathcal{H}$ and a two subsets $X, Y \subseteq V(\mathcal{H})$ such that $X \cap Y = \emptyset$.
Question: Does there exist a minimal hitting set S of $\mathcal{H}$ such that $X \subseteq S \subseteq V(\mathcal{H}) \setminus Y$?

Problem Ext-MMHS was defined by Bläsius et al. [5]. We also define Simple-Ext-MMHS as the special case of the Ext-MMHS where $Y = \emptyset$.

The last problem we define here is the "max-min" version of Dominating Set.

Upper Dominating Set (Up-Dom)
Input: A graph G and an integer k.
Question: Does G contain a minimal dominating set of size at least k?

Tree decompositions and treewidth. A *tree decomposition* of a graph G is a pair $\mathcal{D} = (T, \mathcal{B})$, where T is a tree and $\mathcal{B} = \{X^w \mid w \in V(T)\}$ is a collection of subsets of $V(G)$, called *bags*, such that:

- $\bigcup_{w \in V(T)} X^w = V(G)$,
- for every edge $\{u, v\} \in E$, there is a $w \in V(T)$ such that $\{u, v\} \subseteq X^w$, and
- for every $\{x, y, z\} \subseteq V(T)$ such that z lies on the unique path between x and y in T, $X^x \cap X^y \subseteq X^z$.

We call the vertices of T *nodes* of $\mathcal{D}$ and the sets in $\mathcal{B}$ *bags* of $\mathcal{D}$. The *width* of a tree decomposition $\mathcal{D} = (T, \mathcal{B})$ is $\max_{w \in V(T)} |X^w| - 1$. The *treewidth* of a graph G, denoted by $\mathsf{tw}(G)$, is the smallest integer t such that there exists a tree decomposition of G of width at most t. We need to introduce nice tree decompositions, which will make the presentation of the algorithm of Section 4 much simpler.

Nice tree decompositions. Let $\mathcal{D} = (T, \mathcal{B})$ be a rooted tree decomposition of G (meaning that T has a special vertex r called the *root*). As T is rooted, we naturally define an ancestor relation among bags, and say that $X^{w'}$ is a *descendant* of X^w if the vertex set of the unique simple path in T from r to w' contains w. In particular, every node w is a descendant of itself. For every $w \in V(T)$, we define $G_{X^w} = G[\bigcup\{X^{w'} \mid X^{w'}$ is a descendant of X^w in $T\}]$.

Such a rooted decomposition is called a *nice tree decomposition* of G if the following conditions hold:

- $X^r = \emptyset$,
- every node of T has at most two children in T,
- for every leaf $\ell \in V(T)$, $X^\ell = \emptyset$. Each such a node ℓ is called a *leaf node*,
- if $w \in V(T)$ has exactly one child w', then either
  - $X^w = X^{w'} \cup \{v\}$ for some $v \notin X^{w'}$. Each such a node is called an *introduce node*,
  - $X^w = X^{w'} \setminus \{v\}$ for some $v \in X^{w'}$. Each uch a node is called a *forget node*, and
- if $w \in V(T)$ has exactly two children w_L and w_R, then $X^w = X^{w_L} = X^{w_R}$. Each such a node is called a *join node*.

In the case of a join node, notice that there is no edge in G_{X^w} between $V(G_{X^{w_L}}) \setminus X^w$ and $V(G_{X^{w_R}}) \setminus X^w$. Given a tree decomposition of a graph G, it is possible to transform it in polynomial time into a *nice* one of the same width [24].

For the sake of simplicity of the (already quite heavy) notation used in the dynamic programming algorithm of Section 4, we will drop the vertices of $V(T)$ from the notation of bags defined above. Therefore, in the case of an introduce or forget node, the bag X^w and its child $X^{w'}$ will be denoted X and X^C, respectively, and in the case of a join node, the bag X^w and its children X^{w_L} and X^{w_R} will be denoted X, X^L, and X^R respectively.

Basic properties. We now state some basic properties of the considered problems that will be used later. The ones concerning minimal blocking sets have been already (explicitly or implicitly) observed in [21], but for the sake of completeness we prove all of them here.

▶ **Lemma 4.** *The following properties hold.*

1. *For every graph G, $B \subseteq V(G)$ is an* mbs *of G if and only if B is a* bs *of G and, for every $v \in B$, there is a* mis *I_v of G such that $I_v \cap B = \{v\}$.*
2. *For every hypergraph $\mathcal{H}$, $B \subseteq V(\mathcal{H})$ is an minimal hitting set of $\mathcal{H}$ if and only if B is a hitting set of $\mathcal{H}$ and, for every $v \in B$, there is a hyperedge H_v of $\mathcal{H}$ such that $H_v \cap B = \{v\}$.*
3. *For every graph G, there exists a unique* mis *in G if and only if* $\mathsf{mmbs}(G) = 1$.

Proof: *Property 1.* For the forward implication, consider an mbs B of G. As B is minimal, for every $v \in B$, there exists a mis I_v such that $I_v \cap \{B \setminus \{v\}\} = \emptyset$. As B is a bs, $I_v \cap B \neq \emptyset$, implying $I_v \cap B = \{v\}$. The backward implication is immediate. The proof of Property 2 is almost the same.

Property 3. For the forward implication, let I be the unique mis of G, B be a bs of G, and $v \in B \cap I$. If $|B| \geq 2$ then B is not minimal as $\{v\}$ is still a bs. Let us now prove the contrapositive of the backward implication. Suppose that G contains two distinct mis I_1 and I_2. Since I_1 and I_2 have the same cardinality and are distinct, there is $v_1 \in I_1 \setminus I_2$ and there is $v_2 \in I_2 \setminus I_1$. Note that $\{v_1, v_2\}$ is a minimal blocking set of G, thus $\mathsf{mmbs}(G) \geq 2$.

□

3 Parameterization by α and β

In this section we establish the results summarized in Table 1 about the parameterized complexity of MMBS and MMHS under several parameterizations depending on α and β. We present the negative and the positive results in Section 3.1 and Section 3.2, respectively.

3.1 Hardness results

It is natural to ask, for a graph G, whether computing $\alpha(G)$ can help toward computing $\mathsf{mmbs}(G)$, and vice-versa. In fact, the parameters α and mmbs are linked by the duality relation discussed in what follows.

Given a ground set S, a *clutter* is a family $\mathcal{A}$ of subsets of S such that no set $A_1 \in \mathcal{A}$ contains another set $A_2 \in \mathcal{A}$. Given a clutter $\mathcal{A}$, the family of *blocking sets* of $\mathcal{A}$, denoted by $b(\mathcal{A})$, is the set of minimal subsets B of S such that B intersects every set $A \in \mathcal{A}$. Notice that $b(\mathcal{A})$ is a clutter, and thus $b(b(\mathcal{A}))$ is well-defined. The following theorem provides a duality relation and can be found, for instance, in [4].

▶ **Theorem 5.** $b(b(\mathcal{A})) = \mathcal{A}$.

If we apply Theorem 5 to our setting, namely with $\mathcal{A}$ being the set of all mis of a graph G, we get that $b(\mathcal{A})$ is the set of all mbs, and that the set of minimal sets intersecting all the sets in $b(\mathcal{A})$ is the set of all mis. Even if this theorem gives a relation between mbs and mis, it seems, to the best of our knowledge, that it does not provide a way to compute $\alpha(G)$ from $\mathsf{mmbs}(G)$, or $\mathsf{mmbs}(G)$ from $\alpha(G)$.

Let us start with the easy direction.

▶ **Property 2.** *Let $\mathcal{F}$ be a hereditary graph class. If the problem of computing an* mbs *(not necessarily maximum) is polynomial-time solvable on $\mathcal{F}$, then* IS *is polynomial-time solvable on $\mathcal{F}$. This implies that if* MMBS *is polynomial-time solvable on $\mathcal{F}$, then* IS *is polynomial-time solvable on $\mathcal{F}$.*

Proof: Suppose that we have an algorithm that, given a graph $G \in \mathcal{F}$, outputs in polynomial time an mbs B of G. According to Lemma 4 (Property 1), for every $v \in B$ there exists a mis I_v such that $I_v \cap B = \{v\}$, implying that $\alpha(G \setminus B) = \alpha(G) - 1$ as B is a blocking set. As $G \setminus B \in \mathcal{F}$ because $\mathcal{F}$ is hereditary, we can repeat the same argument to $G \setminus B$, stopping when we obtain an empty graph. It follows that $\alpha(G)$ is equal to the number of iterations of this procedure. ◻

Let us now show that there is no hope to get the same kind of property in the backward direction. We point out a related result in [21] showing that there is a graph class $\mathcal{F}$ where $\mathsf{mmbs}(\mathcal{F}) = 1$, as there is a unique mis for any $G \in \mathcal{F}$ (see Property 3 of Lemma 4), but IS is not polynomial-time solvable unless $\mathsf{NP} = \mathsf{RP}$. This result is obtained through a reduction from Unique-SAT and guarantees that if the original instance is a yes-instance, then there is a unique mis of size k, and otherwise a unique mis of size $k-1$. In the following result, the situation is different, as we target a complexity result for MMBS, and not for IS. For a graph class $\mathcal{F}$, let $\alpha(\mathcal{F}) = \sup_{G \in \mathcal{F}} \alpha(G)$.

▶ **Proposition 6.** *There exist*

- *a hereditary graph class $\mathcal{F}$ where $\alpha(\mathcal{F}) \leq 2$ and on which* MMBS *is* NP*-hard (implying that* 2-MMBS *is* NP*-hard), and*
- *a graph class $\mathcal{F}$ where* IS *is polynomial-time solvable, and* MMBS/α *is* $\mathsf{W}[1]$*-hard, even the particular case of deciding, given an input graph G, whether* $\mathsf{mmbs}(G) > 1$*. This implies that* MMBS/β *is para-*NP*-hard, and that* MMBS/$(\alpha+\beta)$ *is* $\mathsf{W}[1]$*-hard.*

Proof: It is known that IS remains NP-hard on triangle-free graphs. Indeed, Poljak [26] observed that if F is the graph obtained from G by subdividing every edge of G twice, then $\alpha(F) = \alpha(G) + m(G)$. Since F is triangle-free, the result follows.

Boria et al. [8, Theorem 1] presented a reduction from IS to MMVC to deduce an inapproximability result for MMVC. Their reduction builds, from an instance (G, k) of IS, an instance (H, k') of MMVC by adding to G $n(G) + 1$ new pendant vertices at each vertex $v \in V(G)$, i.e. adding $n(G)+1$ vertices of degree one whose unique neighbor in H is v, in order to obtain H. Then, they observe that G has an independent set S of cardinality k if, and only if, $V(H) \setminus S$ is a minimal vertex cover of H, i.e. $k' = n(G) - k + k(n(G) + 1) = (k-1)n(G)$.

The addition of a vertex of degree one does not introduce any triangle in a graph. Thus, by applying the reduction of Boria et al. to an instance of IS restricted to triangle-free graphs, we may deduce that MMVC is NP-hard on triangle-free graphs.

To prove our first statement, we reduce from MMVC on triangle-free graphs, and given an input G of MMVC, we define our input G' of MMBS as the complement of G (that is, the graph obtained from G by swapping edges and non-edges). We may also assume that G contains at least one edge. Observe that as G is triangle-free and G contains at least one edge, $\alpha(G') = 2$. Moreover, there is a bijection between edges of G and mis of G'. This implies that for every subset $B \subseteq V(G)$, B is a vertex cover of G if and only if B is a bs of G', and thus that B is a minimal vertex cover of G if and only if B is an mbs of G'.

To prove our second statement, we reduce from the Multicolored Independent Set (CIS) problem. It is known that CIS$/k$ is W[1]-hard [14] . Given an input (G, k) of CIS, where $V(G)$ is partitioned into k cliques $\{V_i \mid i \in [k]\}$, let G_1 be a copy of G and let G_2 be the graph composed of an is of size k. We define G' as the graph obtained by taking the disjoint union of G_1 and G_2, and adding all edges between $V(G_1)$ and $V(G_2)$. Observe that $|V(G')| = |V(G)| + k$, that $\alpha(G') \leq k$ as $\alpha(G_i) \leq k$, and $\alpha(G') = k$ as $V(G_2)$ is an is.

If (G, k) is a yes-instance, there are two distinct (even disjoint) mis I_i in G': we can define $I_1 \subseteq V(G_1)$ as an is of size k in G, and $I_2 = V(G_2)$. This implies by Lemma 4 (Property 3) that $\mathsf{mmbs}(G') \geq 2$.

Conversely, if (G, k) is a no-instance, the unique mis of G' is $V(G_2)$, implying by Lemma 4 (Property 3) that $\mathsf{mmbs}(G') = 1$. Finally, this is a parameterized reduction as $\alpha(G') \leq k$, and IS is polynomial-time solvable restricted to the family of graphs produced by the reduction. □

Let us now turn to lower bounds for MMHS$/\beta$. It is known that MMHS$/\beta$ is W[1]-hard [2] and that, unless the ETH fails, Simple-Ext-MMHS cannot be solved in time $f(|X|) \cdot (n+m)^{o(|X|)}$ for any computable function $f : \mathbb{N} \to \mathbb{N}$, where n and m are the number of vertices and hyperedges of the input hypergraph, respectively [5]. In our next theorem we prove the same lower bound for Up-Dom, transferring the result to MMHS as well (recall that Up-Dom is a special case of MMHS).

Let us mention that the reduction for MMHS$/\beta$ of Bazgan et al. [2] is a reduction from Multicolored Independent Set parameterized by k, showing that, in fact, Up-Dom is W[1]-hard parameterized by the solution size, where the parameter of the Up-Dom instance is $\mathcal{O}(k^2)$. While being indeed a parameterized reduction, it only implies that, unless the ETH fails, Up-Dom cannot be solved in time $f(k) \cdot (n+m)^{o(\sqrt{k})}$ for any computable function $f : \mathbb{N} \to \mathbb{N}$. We also mention that very recently and independently from our work, Theorem 7 has also been proved by Dublois et al. [18], by using a reduction quite similar to ours.

▶ **Theorem 7.** *Unless the* ETH *fails, the* Upper Dominating Set *problem cannot be solved in time $f(k) \cdot |V(G)|^{o(k)}$ for any computable function $f : \mathbb{N} \to \mathbb{N}$.*

Proof: Chen et al. [11] proved that, unless the ETH fails, the k-Clique problem, i.e. deciding whether a given graph has a clique of size at least k, cannot be solved in time $f(k) \cdot n^{o(k)}$ on n-vertex graphs for any computable function $f : \mathbb{N} \to \mathbb{N}$.

In the Multicolored k-Independent Set problem, we are given a graph G and an integer parameter k, such that $V(G)$ is partitioned into k sets $V_1 \uplus \cdots \uplus V_k$, and the question is whether G contains an independent set containing exactly one vertex in V_i, for $i \in [k]$. There is a simple parameterized reduction from k-Clique to the Multicolored k'-Independent Set problem parameterized by k', namely CIS$/k'$, with linear dependency on the parameter, i.e. k' is linear in k [14]. Thus, the result of Chen et al. [11] implies that the CIS cannot be solved in time $f(k) \cdot n^{o(k)}$ on n-vertex graphs for any computable function $f : \mathbb{N} \to \mathbb{N}$.

We present a parameterized reduction from CIS to Up-Dom that, given an instance (G, k) of CIS, creates in polynomial time a graph G' that contains a minimal dominating set of size at least $3k$ if and only if G contains a multicolored is of size k. By the above discussion, such a reduction concludes the proof of the theorem.

Given G, with $V(G) = V_1 \uplus \cdots \uplus V_k$, for every color $i \in [k]$ we add to G' three copies A_i, B_i, C_i of V_i, and let $U_i = A_i \cup B_i \cup C_i$ be their union. We denote $A = \bigcup_{i \in [k]} A_i$,

$B = \bigcup_{i\in[k]} B_i$, $C = \bigcup_{i\in[k]} C_i$, and, for a vertex $v \in V(G)$, we denote by v_A, v_B, v_C its corresponding copy in A, B, C, respectively. For every $i \in [k]$, the set U_i induces, in G', a clique minus the triangles $\{(v_A, v_B, v_C) \mid v \in V_i\}$. That is, within the same color i, every vertex is adjacent to all other vertices except for its two other copies. For every edge $\{u, v\} \in E(G)$ such that $u \in V_i$ and $v \in V_j$ with $i \neq j$, we add to G' the edges $\{u_A, v_B\}$ and $\{u_B, v_A\}$. This concludes the construction of G'. We claim that G contains a multicolored is of size k if and only if G' that contains a minimal dominating of size at least $3k$.

Let first $S \subseteq V(G)$ be a multicolored is of size k. Let $D \subseteq V(G')$ contain, for every vertex $v \in S$, its three copies v_A, v_B, v_C. Note that $|D| = 3k$. We claim that D is a minimal dominating set of G'. Since D contains a vertex in each of the $3k$ cliques into which $V(G')$ is partitioned, D is clearly a dominating set. Consider a vertex $v_A \in D \cap A$ (the case $v_B \in D \cap B$ is symmetric). Then $D \setminus \{v_A\}$ is not a dominating set, since by the hypothesis that S is an is in G, no vertex in $D \setminus \{v_A\}$ is adjacent to v_A. Consider now a vertex $v_C \in D \cap C$, with $v \in V_i$. Then $D \setminus \{v_C\}$ is not a dominating set either, as $D \cap (A_i \cup B_i) = \{v_A, v_B\}$, and none of v_A and v_B is adjacent to v_C. Hence, D is a minimal dominating set of G' and we are done.

Conversely, let $D \subseteq V(G')$ be a minimal dominating set with $|D| \geq 3k$.

▷ Claim 8. For every $i \in [k]$, $|D \cap A_i| \leq 1$ and $|D \cap B_i| \leq 1$.

Proof of the claim: We say that an index $i \in [k]$ is *abnormal* if $|D \cap A_i| \geq 2$ or $|D \cap B_i| \geq 2$ (or both), and *normal* otherwise. We will construct a set $D' \subseteq V(G')$ with $|D'| = |D|$ such that if i is normal, $|D' \cap U_i| \leq 3$, and if i is abnormal, $|D' \cap U_i| \leq 2$. Hence, if there exists an abnormal index, it holds that $|D| = |D'| < 3k$, contradicting the hypothesis that $|D| \geq 3k$. We now proceed to the construction of D', which is not required to be a dominating set of G'. We start with $D' = D$, and we update D' as described below.

For every abnormal index i, we do the following. Since $|D \cap A_i| \geq 2$ or $|D \cap B_i| \geq 2$, by construction of G' we have that $D \cap (A_i \cup B_i)$ dominates U_i, and since $N(C_i) = A_i \cup B_i$, necessarily $|D \cap C_i| = 0$, as otherwise D would not be minimal. If $|D \cap U_i| = 2$ we do nothing, as we already have that $|D' \cap U_i| = |D \cap U_i| \leq 2$. Assume henceforth that $|D \cap U_i| = |D \cap (A_i \cup B_i)| \geq 3$.

To simplify the presentation, suppose that $|D \cap A_i| \geq |D \cap B_i|$, the other case being symmetric. Hence, we have that $|D \cap A_i| \geq 2$. Note that any two vertices in the set $D \cap A_i$, say u_A and v_A, dominate the whole set U_i. Hence, since D is a minimal dominating set of G', for every other vertex $w_A \in (D \cap A_i) \setminus \{u_A, v_A\}$ (resp. $w'_B \in D \cap B_i$) there must exist an index $j \neq i$ (resp. $\ell \neq i$) and a vertex $z_B \in B_j$ (resp. $z'_A \in A_\ell$) not in D and dominated only by w_A (resp. w'_B), that is, with $z_B \notin D$ (resp. $z'_A \notin D$) and $N_D(z_B) = \{w_A\}$ (resp. $N_D(z'_A) = \{w'_B\}$); see Figure 1(a) for an illustration, where the vertices in D are depicted in red. Note that such an index j (resp. ℓ) is necessarily normal, as otherwise vertex z_B (resp. z'_A) would be already dominated within U_j (resp. U_ℓ). Note also that, for the same reason, $D \cap B_j = \emptyset$ (resp. $D \cap A_\ell = \emptyset$). For each such a vertex $w_A \in D$ (resp. $w'_B \in D$), we remove vertex w_A (resp. w'_B) from D' and we add vertex z_B (resp. z'_A) to D'; see Figure 1(b) for an illustration, where the vertices in D' are depicted in red. We say that vertex $z_B \in B_j$ (resp. $z'_A \in A_\ell$) is a *sink*. This concludes the construction of D'. It just remains to verify that the claimed properties of D' are satisfied.

By construction, we clearly have that $|D'| = |D|$. Note that, if i is an abnormal index as in the above paragraph, then $A_i \cup B_i$ cannot contain any sink since all the vertices of U_i are already dominated by $D \cap U_i$. Hence, no vertex is added to $D' \cap U_i$ and it holds that $D' \cap U_i = \{u_A, v_A\}$, so we indeed have that $|D' \cap U_i| \leq 2$.

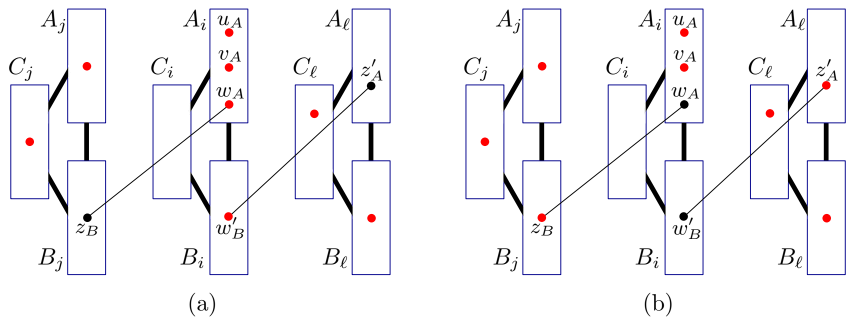

Figure 1 Configuration in the proof of Claim 8. Index i is abnormal, while indices j and ℓ are normal. Vertices z_B and z'_A are sinks. (a) The vertices in D are depicted in red. (b) The vertices in D' are depicted in red.

It remains to verify that, if j is a normal index, then $|D' \cap U_j| \leq 3$. Since the vertices in C_j have neighbors only in U_j, necessarily $|D \cap U_j| \geq 1$. Hence, at most one vertex in A_j and at most one vertex in B_j are not dominated by the vertices in $D \cap U_j$. Thus, each of A_j and B_j contains at most one sink. We distinguish three cases according to the number of sinks in $A_j \cup B_j$.

Suppose first that $A_j \cup B_j$ contains no sink. In this case, note that $|D' \cap U_j| = |D \cap U_j|$, by the definition of D'. Recall that, since j is normal, then $|D \cap A_j| \leq 1$ and $|D \cap B_j| \leq 1$. Since D is a minimal dominating set and $N(C_j) \subseteq A_j \cup B_j$, note that $|D \cap C_j| \leq 2$, because two vertices in C_j suffice to dominate all vertices $A_j \cup B_j$. However, it is not possible to have $|D \cap C_j| = 2$, $|D \cap A_j| = 1$, and $|D \cap B_j| = 1$ because of the minimality of D, as one vertex of C_j could be removed. Thus, $|D' \cap U_j| = |D \cap U_j| \leq 3$.

Suppose now that $A_j \cup B_j$ contains exactly one sink, so we have that $|D' \cap U_i| = |D \cap U_i| + 1$. Suppose without loss of generality that the sink is a vertex $z_A \in A_j$, so we have $|D \cap A_j| = 0$. Since z_A is not dominated by $D \cap U_i$, necessarily $|D \cap B_j| \leq 1$ and $|D \cap C_j| \leq 1$, so $|D \cap U_i| \leq 2$ and $|D' \cap U_i| \leq 3$.

Finally, suppose that $A_j \cup B_j$ contains two sinks $z_A \in A_j$ and $z'_B \in B_j$, so we have that $|D' \cap U_i| = |D \cap U_i| + 2$, $|D \cap A_j| = 0$, and $|D \cap B_j| = 0$. Also, since none of z_A and z'_B can be dominated by $D \cap U_j$, necessarily $z = z'$ and $D \cap C_j = \{z_C\}$, so $|D \cap C_j| = 1$. Thus, $|D \cap U_i| \leq 1$ and $|D' \cap U_i| \leq 3$, and the claim follows. $\diamond$

▷ Claim 9. For every $i \in [k]$, $|D \cap U_i| \leq 3$.

Proof of the claim: Suppose for contradiction that there exists $i \in [k]$ such that $|D \cap U_i| \geq 4$. By Claim 8, necessarily $|D \cap C_i| \geq 2$. If $|D \cap C_i| \geq 3$, then deleting all but any two vertices in $D \cap C_i$ results in a proper subset of D that is still a dominating set of G', contradicting the minimality of D. Hence $|D \cap C_i| = 2$, $|D \cap A_i| = 1$, and $|D \cap B_i| = 1$. Let $u_A \in D \cap A_i$ and let $v_C, w_C \in D \cap C_i$. At least one among v and w, say v, is not equal to u. Then the set $D \setminus \{w_C\}$ is still a dominating set of G', contradicting again the minimality of D. $\diamond$

▷ Claim 10. For every $i \in [k]$, $|D \cap A_i| = |D \cap B_i| = |D \cap C_i| = 1$.

Proof of the claim: Since by hypothesis we have that $|D| \geq 3k$ and by Claim 9 it holds that $|D \cap U_i| \leq 3$ for every $i \in [k]$, necessarily $|D \cap U_i| = 3$ for every $i \in [k]$. We claim that, for every $i \in [k]$, $|D \cap C_i| = 1$. Suppose first for contradiction that there exists $i \in [k]$ such that $|D \cap C_i| = 3$. Then $|D \cap A_i| = |D \cap B_i| = 0$, and any two of the three vertices in $|D \cap C_i|$

dominate U_i (recall that all the neighbors of the vertices in C_i are in U_i), contradicting the minimality of D. Assume now, again for contradiction, that there exists $i \in [k]$ such that $|D \cap C_i| = 2$. Suppose without loss of generality that $|D \cap A_i| = 1$ and $|D \cap B_i| = 0$, and let $D \cap A_i = \{u_A\}$ and $D \cap C_i = \{v_C, w_C\}$. At least one vertex among v and w is different from u, say v. Then $D \setminus \{w_C\}$ is still a dominating set of G', contradicting the minimality of D. Thus, for every $i \in [k]$, $|D \cap C_i| = 1$. Therefore, since, for every $i \in [k]$, $|D \cap U_i| = 3$ and, by Claim 8 we have that $|D \cap A_i| \leq 1$ and $|D \cap B_i| \leq 1$, we conclude that $|D \cap A_i| = |D \cap B_i| = |D \cap C_i| = 1$. $\diamond$

We proceed to define from D a multicolored is $S \subseteq V(G)$ with $|S| = k$. Consider an arbitrary index $i \in [k]$. By Claim 10, $D \cap A_i = \{u_A\}$, $D \cap B_i = \{v_B\}$, and $D \cap C_i = \{w_C\}$. Note that if $u \neq v$, then u_A and v_B would dominate the whole set U_i and w_C could be removed from D, contradicting its minimality. Thus, we have that $u = v$. We define $S \cap V_i = \{v\}$. It remains to verify that S is indeed an is of G. Consider $u, v \in S$ with $u \in V_i$ and $v \in V_j$. If $\{u, v\} \in E(G)$ then $D \setminus \{u_A\}$ would still be a dominating set of G'. Indeed, vertex u_A would be dominated by v_B, and the other vertices in A_i would still be dominated by u_B. Thus, $\{u, v\} \notin E(G)$ and S is indeed a multicolored is in G. $\square$

Theorem 7 immediately yields the following corollary for MMHS.

▶ **Corollary 11.** *Unless the* ETH *fails,* MMHS *cannot be solved in time* $f(\beta) \cdot (|V(\mathcal{H})| + |E(\mathcal{H})|)^{o(\beta)}$ *for any computable function* $f : \mathbb{N} \to \mathbb{N}$.

Proof: As mentioned in Section 1 – Problems related to computing mmbs – recall that Up-Dom is a particular case of MMHS when we consider each closed neighborhood of each vertex $v \in V(G)$ of an instance (G, k) of Up-Dom to be a hyperedge H of an instance $(\mathcal{H}, k)$ of MMHS having $V(G)$ as vertex set of $\mathcal{H}$. $\square$

3.2 Positive results

Let us now turn to positive results, and consider a graph class where IS can be solved in polynomial time. As according to Proposition 6 we cannot hope to solve MMBS in polynomial time, we consider the parameterized complexity of the MMBS problem. The first result shows the crucial difference between the problems of, given a graph G and a positive integer β, deciding whether $\mathsf{mmbs}(G) = \beta$ and deciding whether $\mathsf{mmbs}(G) \geq \beta$. In any maximization problem, the first property implies the second one, but the backward implication is not always true. In particular, for MMBS, $\mathsf{mmbs}(G) \geq \beta$ does *not* imply that G contains an mbs of size *exactly* β, and this is informally what makes the inequality version harder.

As observed in [20] or in [15, Proposition 1], deciding whether there exists a minimal hitting set of size exactly β, or at most β, in a hypergraph with hyperedges of size at most α can be trivially decided by a search-tree in time $\mathcal{O}^*(\alpha^\beta)$. However, we cannot use this result directly, as a reduction to MMHS would require time $n^{\alpha(G)}$ to generate the hyperedges, and thus we have to define an ad-hoc algorithm, which is also based on branching.

▶ **Proposition 12.** *Let $\mathcal{F}$ be a hereditary graph class on which* IS *is polynomial-time solvable. Then* $\text{MMBS}^=/(\alpha+\beta)$ *and* $\text{MMBS}^{\leq}/(\alpha+\beta)$ *are* FPT *restricted to input graphs in $\mathcal{F}$. More precisely, they can both be solved in time $\mathcal{O}^*(\alpha(G)^\beta)$.*

Proof: We only prove the result for $\text{MMBS}^=$, as it directly implies the result for $\text{MMBS}^{\leq}$. Consider an input graph $G \in \mathcal{F}$. Let us define an algorithm $A(X)$ that, given a set $X \subseteq V(G)$

with $|X| \leq \beta$, answers "yes" if and only if there exists an mbs B of G such that $X \subseteq B$ and $|B| = \beta$, in which case we say that X is a yes-*set*. The algorithm starts with $X = \emptyset$, and calls itself recursively for a larger set X obtained from branching on vertices of a mis of $G \setminus X$, as detailed below. Note that if $|X| = \beta$, then $A(X)$ answers "yes" if and only if X is an mbs. This can be checked in polynomial time, as it is equivalent to the properties that $\alpha(G \setminus X) < \alpha(G)$ and $\alpha(G \setminus (X \setminus \{v\})) = \alpha(G)$ for every $v \in X$, and as we assumed that IS is polynomial-time solvable in $\mathcal{F}$, and $\mathcal{F}$ is hereditary. Let us now consider the cases where $0 \leq |X| < \beta$. If $\alpha(G \setminus X) < \alpha(G)$, then we answer "no" as X is already a bs, and thus no superset $X' \supsetneq X$ can be an mbs. Otherwise, we have that $\alpha(G \setminus X) = \alpha(G)$. Since $G \setminus X \in \mathcal{F}$ as $\mathcal{F}$ is hereditary and IS is polynomial-time solvable on $\mathcal{F}$, we can compute in polynomial time a mis I of $G \setminus X$. Indeed, note that we can construct a subset S corresponding to a mis of a graph G by knowing an algorithm to the decision problem IS as follows: first compute $\alpha(G)$ by linearly checking the $|V(G)|$ possible values. For every vertex $v \in V(G)$, if $\alpha(G - v) = \alpha(G)$ then remove v from G and iterate; otherwise, add v to S, remove v and its neighborhood from G and iterate. Observe that I is also a mis of G, and that $I \cap X = \emptyset$. In this case, $A(X)$ returns $\bigvee_{v \in I} A(X \cup \{v\})$.

Let us now prove the correctness of this latter case. Suppose first that X is a yes-set, and let B be an mbs in G of size β such that $X \subseteq B$. As B is a bs, there exists $v \in I \cap B$. As $I \cap X = \emptyset$, we get $v \notin X$, and thus $X \cup \{v\}$ is a yes-set and $A(X)$ returns "yes". We prove the other direction by reverse induction on $|X|$, the case $|X| = \beta$ being correct as discussed above. Consider a set X with $|X| < \beta$, and suppose inductively that the claimed property is correct for sets of size $|X| + 1$. Thus, if $A(X \cup \{v\})$ returns "yes" for some $v \in I$, then by induction $(X \cup \{v\})$ is a yes-set, implying by definition of A that X is also a yes-set.

Let us finally discuss the running time of the algorithm, given an input graph G. Starting with $X = \emptyset$, for every set X the algorithm performs a polynomial number of operations, and then branches on a set I of size at most $\alpha(G)$, as such a set I is always a mis of a subgraph of G. As the depth of the branching tree corresponding to the algorithm is at most β, the claimed running time follows. ◻

First note that the hypothesis "IS is polynomial-time solvable" in Proposition 12 could be relaxed to "IS is FPT when parameterized by the natural parameter", but the time complexity for solving MMBS$^{=}/(\alpha + \beta)$ and MMBS$^{\leq}/(\alpha + \beta)$ would be a function of the time complexity of such an FPT algorithm. Observe also that, unless FPT $=$ W[1], we cannot obtain results similar to Proposition 12 to decide whether $\mathsf{mmbs}(G) \geq \beta$, and even in time $\mathcal{O}^*(f(\alpha(G), \beta))$ for any computable function f, as it would imply that MMBS$/(\alpha + \beta)$ is FPT, contradicting the fact that MMBS$/\beta$ is para-NP-hard by Proposition 6. Thus, we need consider a stronger assumption than assuming that IS is polynomial-time solvable on $\mathcal{F}$. Namely, in what follows we consider the α-MMBS problem, that is, the case where α is fixed. Recall that, according to Proposition 6, even 2-MMBS remains NP-hard, motivating the study of the parameterized complexity of α-MMBS.

▶ **Proposition 13.** *For every fixed positive integer α, α-MMBS$/\beta$ and α-MMHS$/\beta$ are* FPT. *More precisely, both problems can be solved in time $\mathcal{O}^*(\alpha^\beta)$.*

Proof: Given an input graph G of α-MMBS, we compute in time $\mathcal{O}^*(n^\alpha)$ the hypergraph $\mathcal{H}$ where $V(\mathcal{H}) = V(G)$, and H is a hyperedge in $\mathcal{H}$ if and only if H is a mis in G. By definition of α-MMBS, all hyperedges of $\mathcal{H}$ have size exactly α, and for every $B \subseteq V(G)$, B is an mbs in G if and only if B is a minimal hitting set in $\mathcal{H}$. Then, according to [15, Lemma 6], as α is fixed we can decide whether there is a minimal hitting set of $\mathcal{H}$ of size at least β in time $\mathcal{O}^*(\alpha^\beta)$. ◻

We point out that in both Proposition 13 and [15, Lemma 6], in order to decide whether there is a minimal hitting set of $\mathcal{H}$ of size at least β, there is a term $|V(\mathcal{H})|^{f(\alpha)}$ hidden inside the $\mathcal{O}^*$-notation. This means that [15, Lemma 6] does not imply that MMHS/$(\alpha+\beta)$ is FPT (recall that function α in the parameterization of MMHS denotes the size of a largest hyperedge of $\mathcal{H}$). However, according to the following two propositions, it turns out that MMHS/$(\alpha+\beta)$ is indeed FPT. This highlights a difference between MMHS and MMBS, as according to Proposition 6 MMBS/$(\alpha+\beta)$ is unlikely to be FPT.

Let us start with a kernelization result, using the well-known notion of sunflower.

▶ **Definition 14.** *Let $\beta \in \mathbb{N}$. Given a hypergraph $\mathcal{H}$, a* sunflower *in $\mathcal{H}$ with β* petals *and* core *$C \subseteq V(\mathcal{H})$ is a collection of β hyperedges $H_1, \ldots, H_\beta$ of $\mathcal{H}$ such that $H_i \cap H_j = C$ for all $i, j \in [\beta]$, $i \neq j$, and for every $i \in [\beta]$ the so-called petal $H_i \setminus C$ is not empty. We say that a function $s : \mathbb{N}^2 \to \mathbb{N}$ is a* sunflower function *if, for every hypergraph $\mathcal{H}$ whose hyperedges have size at most α, if $|E(\mathcal{H})| > s(\alpha, \beta)$ then $\mathcal{H}$ admits a sunflower with β petals.*

It is known (see for instance [14]) that $s_1(\alpha, \beta) := (\alpha^2)!\alpha(\beta-1)^\alpha$ is a sunflower function. Even if this is not relevant for our next proposition, where α is fixed, we point out that this bound has recently been improved by Bell et al. [3] to

$$s_2(\alpha, \beta) := (c\beta \cdot \log(\alpha))^\alpha \quad \text{for some positive constant } c. \tag{1}$$

▶ **Lemma 15.** *Let $\beta \in \mathbb{N}$, and let $\mathcal{H}$ be a hypergraph such that no hyperedge is included in another hyperedge. If $\mathcal{H}$ has a sunflower with β petals, then $\mathsf{mmhs}(\mathcal{H}) \geq \beta$.*

Proof: Let $\{H_i \mid i \in [\beta]\}$ be a sunflower of $\mathcal{H}$ with β petals. Let $C = \bigcap_{i \in [\beta]} H_i$ and $S = V(\mathcal{H}) \setminus C$. As there is no $H \in E(\mathcal{H})$ such that $H \subseteq C$, it follows that S is a hitting set of $\mathcal{H}$. Let $S' \subseteq S$ be a minimal hitting set of $\mathcal{H}$. For every $i \in [\beta]$, $H_i \setminus C \neq \emptyset$ by definition of a sunflower, hence S' must contain at least one vertex in $H_i \setminus C$. This implies that $|S'| \geq \beta$, and thus that $\mathsf{mmhs}(H) \geq \beta$. ◻

In the next proposition we derive polynomial kernels when α is fixed. The kernel for α-MMBS follows straightforwardly from the kernel for α-MMHS, but we provide the details as we cannot directly claim that any kernel for α-MMHS implies a kernel for MMBS. For example, removing a hyperedge in α-MMHS cannot necessarily be translated to α-MMBS.

▶ **Proposition 16.** *Let α be a fixed positive integer. Let $s(\alpha, \beta)$ be a sunflower function that is polynomial in β. Then,*

- *α-MMBS/β admits a polynomial kernel with at most $\alpha \cdot s(\alpha, \beta)$ vertices, which can be constructed in time $\mathcal{O}^*(|V(G)|^\alpha)$, and*
- *α-MMHS/β admits a polynomial kernel with at most $\alpha \cdot s(\alpha, \beta)$ vertices, which can be constructed in time $\mathcal{O}^*(|V(\mathcal{H})|)$ (that is, not depending on α).*

Proof: Let us start with α-MMHS. Consider an instance $(\mathcal{H}, \beta)$ of α-MMHS. By Observation 1, recall that every vertex belongs to at least one hyperedge. If $|E(\mathcal{H}')| \leq s(\alpha, \beta)$, then we get $|V(\mathcal{H}')| \leq \alpha \cdot |E(\mathcal{H}')| \leq \alpha \cdot s(\alpha, \beta)$ and we are done. Otherwise, as s is a sunflower function, it follows that $\mathcal{H}'$ contains a sunflower with β petals (by Property 1, recall that we assume that our instances for MMHS have no pair of hyperedges such that one is a proper subset of another). According to Lemma 15, we get that $(\mathcal{H}, \beta)$ is a yes-instance.

Let (G, β) be an instance of α-MMBS. In time $\mathcal{O}^*(|V(G)|^\alpha)$ we can compute an equivalent instance $(\mathcal{H}, \beta)$ of α-MMHS by creating a hyperedge for every mis of G. Although there might be a vertex that belongs to no mis of G, by Observation 1, we may assume that $\mathcal{H}$ has no vertex that does not belong to any hyperedge, by removing such vertices if they exist. As remarked before, such vertices cannot be part of any minimal hitting set of $\mathcal{H}$. As all hyperedges have size exactly $\alpha(G)$, no hyperedge can be included in another hyperedge. Again, if $|E(\mathcal{H}')| \leq s(\alpha,\beta)$, then we get $|V(\mathcal{H}')| \leq \alpha \cdot |E(\mathcal{H}')| \leq \alpha \cdot s(\alpha,\beta)$ and we are done. Otherwise, as s is a sunflower function, it follows that $\mathcal{H}'$ contains a sunflower with β petals. According to Lemma 15, we get that $(\mathcal{H},\beta)$ is a yes-instance. ◻

Even if Proposition 16 implies that MMHS/$(\alpha+\beta)$ is FPT, the running time obtaining by applying brute force to the kernelized instance is $\mathcal{O}^*(2^{\alpha\cdot s(\alpha,\beta)})$, and thus doubly exponential in α. This motivates the question of obtaining a faster FPT algorithm for MMHS/$(\alpha+\beta)$. We point out that trying to improve the running time by considering separated parameters, instead of the aggregated parameter $\alpha+\beta$, is not possible as MMHS/β is W[1]-hard [2], and MMHS/α is already NP-hard for $\alpha = 2$, as it corresponds to MMVC. A first way to get a faster FPT algorithm is to reduce to the Simple-Ext-MMHS problem.

Please recall that in the Simple-Ext-MMHS problem we are given a hypergraph $\mathcal{H}$ and a subset $X \subseteq V(\mathcal{H})$ and the question is whether $\mathcal{H}$ has a minimal hitting set S such that $X \subseteq S$. Bläsius et al. [5] proved (they actually study the more general Ext-MMHS version) that Simple-Ext-MMHS can be solved in time $\mathcal{O}^*(\lambda^{|X|})$, where $\lambda = \min\left(\frac{|E(\mathcal{H})|}{|X|}, \Delta(\mathcal{H})\right)$ and $\Delta(\mathcal{H}) = \max_{v\in V(\mathcal{H})} |\{H \in E(\mathcal{H}) \mid v \in H\}|$ is the maximum degree of $\mathcal{H}$. Let us denote this algorithm presented by Bläsius et al. [5] as B. For precise details about B, the reader is referred to [5]. Informally, algorithm B just guesses for each $x \in X$ its "private" hyperedge H_x such that $H_x \cap X = \{x\}$, and checks that there is no $H \in E(\mathcal{H})$ such that $H \subseteq (\bigcup_{x\in X} H_x) \setminus X$. Thus, guessing a hyperedge for every $x \in X$ yields the claimed running time. In the next proposition we formalize these ideas, using ideas similar to those in the proof of Proposition 12.

▶ **Proposition 17.** *We can decide an instance $(\mathcal{H},\beta)$ of* MMHS *in time $\mathcal{O}^*((\alpha(\mathcal{H})\cdot\lambda)^\beta)$, where $\lambda = \min\left(\frac{|E(\mathcal{H})|}{\beta}, \Delta(\mathcal{H})\right)$ and $\Delta(\mathcal{H}) = \max_{v\in V(\mathcal{H})} |\{H \in E(\mathcal{H}) \mid v \in H\}|$.*

Proof: Let $(\mathcal{H},\beta)$ be an instance of MMHS. To decide $(\mathcal{H},\beta)$, we actually propose a more general algorithm that decides $(\mathcal{H},\beta)$ while forcing a subset $X \subseteq V(\mathcal{H})$ to be part of a solution. Thus, let us define an algorithm A that, given $\mathcal{H}$, β, and a set $X \subseteq V$ satisfying $|X| \leq \beta$, decides whether there exists a minimal hitting set S of $\mathcal{H}$ such that $X \subseteq S$ and $|S| \geq \beta$. The algorithm A has recursive calls in which the third parameter, i.e. the set X, may change. The algorithm A starts with $X = \emptyset$, i.e. in order to decide $(\mathcal{H},\beta)$ we just output $A(\mathcal{H},\beta,\emptyset)$. Let us describe A, depending on $|X|$.

If $|X| = \beta$, then algorithm A returns $A(\mathcal{H},\beta,X) = B(\mathcal{H},X)$ (recall that B is the algorithm of Bläsius et al. [5] for Simple-Ext-MMHS described above). Note that, in case $|X| = \beta$, we just need to decide whether there is a minimal hitting set extending X, which is exactly what algorithm B does.

Suppose now that $0 \leq |X| < \beta$. If there is no $H \in E(\mathcal{H})$ such that $H \cap X = \emptyset$, then algorithm A outputs "no". Indeed, note that, since $|X| < \beta$, at least one vertex in $V(\mathcal{H}) \setminus X$ needs to be included into a solution S, but then S will certainly be not a *minimal* hitting set if there is no $H \in E(\mathcal{H})$ such that $H \cap X = \emptyset$.

Otherwise, let $H \in E(\mathcal{H})$ such that $H \cap X \neq \emptyset$. In this case $A(\mathcal{H},\beta,X)$ returns $\bigvee_{v\in H} A(\mathcal{H},\beta,X\cup\{v\})$. Since $H \cap X = \emptyset$ and any solution S must be a hitting set, we need to include at least one vertex of H into S. Thus, algorithm A performs brute force by checking all possible cases of addition of a vertex of H to be in S, by adding it to the set X of forced vertices. Note that indeed $\mathcal{H}$ has a minimal hitting set S such that $X \subseteq S$ and $|S| \geq \beta$ if, and only if, $\mathcal{H}$ has a minimal hitting set S' such that $X \cup \{v\} \subseteq S'$ and $|S'| \geq \beta$, for some $v \in H$.

As in each step of algorithm A we branch, in the worst case, on all vertices of a hyperedge H, the running time is bounded by $\mathcal{O}^*(\alpha^\beta \cdot f(\alpha,\beta))$, where $\mathcal{O}^*(f(\alpha,\beta))$ is the running time of algorithm B. Recall that $f(\alpha,\beta) = \mathcal{O}^*(\lambda^\beta)$ and $\lambda = \min\left(\frac{|E(\mathcal{H})|}{\beta}, \Delta(\mathcal{H})\right)$. ◻

▶ **Corollary 18.** *The following claims hold:*

- MMHS/β *is* XP.
- MMHS/$(\alpha+\beta)$ *is* FPT*. More precisely, it can be solved in time $\mathcal{O}^*(\alpha^\beta(c\beta\cdot\log(\alpha))^{\alpha\beta})$, where c is the constant in the sunflower function s_2 of Equation (1).*

Proof: The fact that MMHS/β is XP is a direct consequence of Proposition 17.

To prove that MMHS/$(\alpha+\beta)$ is FPT, one can obtain an FPT algorithm as follows. Given an instance $(\mathcal{H},\beta)$, recall that by Observation 1 $\mathcal{H}$ has no vertex that does not belong to any hyperedge of $\mathcal{H}$. Then, notice that $\Delta(\mathcal{H}') \leq |E(\mathcal{H}')|$. By Property 1, we may also assume that $\mathcal{H}'$ has no pair of hyperedges one being a subset of another. Thus, if $|E(\mathcal{H}')| > s_2(\alpha,\beta)$, then $\mathcal{H}'$ has a sunflower with β petals and it is an "yes" instance thanks to Lemma 15. Otherwise, we have that (recall that s_2 is the sunflower function defined in Equation (1)):

$$\lambda = \min\left(\frac{|E(\mathcal{H}')|}{\beta}, \Delta(\mathcal{H}')\right) \leq \min\left(\frac{|E(\mathcal{H}')|}{\beta}, |E(\mathcal{H}')|\right) \leq \frac{s_2(\alpha,\beta)}{\beta} = \frac{(c\beta\cdot\log(\alpha))^\alpha}{\beta}.$$

Thus, by applying Proposition 17 to $(\mathcal{H}',\beta)$, we get the desired FPT algorithm. ◻

Even if the algorithm of Proposition 17 gives a running time matching the lower bound of Corollary 11 for the dependency on $|E(\mathcal{H})|$, we can get a faster FPT algorithm for MMHS/$(\alpha+\beta)$ using an ad-hoc algorithm that does not reduce to the extension problem. Namely, we present in Theorem 24 an algorithm for MMHS/$(\alpha+\beta)$ running in time $\mathcal{O}^*(2^{\alpha\beta})$. We first need some preliminaries.

▶ **Definition 19.** *Let $\mathcal{H}$ be a hypergraph, let $I \subseteq V(\mathcal{H})$ be an* is *in $\mathcal{H}$, and let $X \subseteq V(\mathcal{H})$. Let*

- *$\mathcal{H}_I$ such that $V(\mathcal{H}_I) = V(\mathcal{H}) \setminus I$, and $E(\mathcal{H}_I) = \{H \setminus I \mid H \in E(\mathcal{H})\}$,*
- *$E^{\bar{X}} = \{H \in E(\mathcal{H}) \mid H \cap X = \emptyset\}$, and*
- *$\mathcal{H}^{\bar{X}}$ such that $V(\mathcal{H}^{\bar{X}}) = V(\mathcal{H})$ and $E(\mathcal{H}^{\bar{X}}) = E^{\bar{X}}$.*

▶ **Lemma 20.** *Let $\mathcal{H}$ be a hypergraph and let $I \subseteq V(\mathcal{H})$ be an* is *in $\mathcal{H}$.*

1. *For every minimal hitting set S of $\mathcal{H}_I$, S is also a minimal hitting set of $\mathcal{H}$. This implies* $\mathsf{mmhs}(\mathcal{H}) \geq \mathsf{mmhs}(\mathcal{H}_I)$.
2. *For every minimal hitting set S^* of $\mathcal{H}$ such that $S^* \cap I = \emptyset$, S^* is also a minimal hitting set of $\mathcal{H}_I$.*

Proof: For the first property, let S be a minimal hitting set of $\mathcal{H}_I$. Let us first prove that S is a hitting set of $\mathcal{H}$. Consider an arbitrary hyperedge $H \in E(\mathcal{H})$. As I is an is, $H \setminus I \neq \emptyset$, and as S is a hitting set of $\mathcal{H}_I$ and $H \setminus I \in E(\mathcal{H}_I)$, we get $S \cap (H \setminus I) \neq \emptyset$. Let us now prove that S is minimal. Consider an arbitrary vertex $v \in S$. By the minimality in $\mathcal{H}_I$, there exists $H \in E(\mathcal{H}_I)$ such that $(S \setminus \{v\}) \cap H = \emptyset$, implying that $(S \setminus \{v\}) \cap (H \cup I) = \emptyset$ as $S \cap I = \emptyset$, where $H \cup I \in E(\mathcal{H})$.

For the second property, let $H' \in E(\mathcal{H}_I)$, where $H' = H \setminus I$, $H \in E(\mathcal{H})$. As S^* is a hitting set of $\mathcal{H}$, $S^* \cap H \neq \emptyset$, and as $S^* \cap I = \emptyset$, we get $S^* \cap H' \neq \emptyset$. Let us now verify the minimality. Consider an arbitrary vertex $v \in S^*$. As S^* is minimal in $\mathcal{H}$, there exists $H \in E(\mathcal{H})$ such that $(S^* \setminus \{v\}) \cap H = \emptyset$, implying $(S^* \setminus \{v\}) \cap (H \setminus I) = \emptyset$. ◻

▶ **Lemma 21.** *Let $\mathcal{H}$ be a hypergraph, let $X \subseteq V(\mathcal{H})$, and let S' be a minimal hitting set of $\mathcal{H}^{\bar{X}}$. There exists a minimal hitting set S of $\mathcal{H}$ such that $S' \subseteq S$.*

Proof: Let $S = S' \cup X$. Observe that S is a hitting set of $\mathcal{H}$. Now, as far as there exists $v \in S \cap X$ such that $S \setminus \{v\}$ is still a hitting set of $\mathcal{H}$, remove v from S. Let S^* be the obtained set, which satisfies $S' \subseteq S^* \subseteq S$, and let us verify that S^* is minimal. For every $v \in S^* \cap X$, by definition of S^* we have that $S^* \setminus \{v\}$ is not a hitting set of $\mathcal{H}$. For every $v \in S^* \cap S'$, as S' is minimal in $\mathcal{H}^{\bar{X}}$, it follows that there exists $H \in E^{\bar{X}}$ such that $(S' \setminus \{v\}) \cap H = \emptyset$. As $H \cap X = \emptyset$, we get $(S^* \setminus \{v\}) \cap H = \emptyset$ as well. ◻

We are now ready to present our FPT algorithm.

▶ **Definition 22.** *Given an instance $(\mathcal{H}, \beta)$ for* MMHS *and $X \subseteq V(\mathcal{H})$, we define algorithm $A^\beta(\mathcal{H}, X)$ as follows:*

- *If $E^{\bar{X}} = \emptyset$,*
  - *if $|X| \geq \beta$ and X is minimal hitting set of $\mathcal{H}$, return "yes".*
  - *Otherwise, return "no".*
- *Otherwise, build S be a minimal hitting set of $\mathcal{H}^{\bar{X}}$.*
  - *If $|S| \geq \beta$, return "yes".*
  - *Otherwise, return $\bigvee_{S_1 \in \mathcal{L}} A^\beta(\mathcal{H}_{S_1}, X \cup (S \setminus S_1))$, where $\mathcal{L} = \{S_1 \subseteq S \mid S_1$ is an* is *of $\mathcal{H}^{\bar{X}}\}$.*

In order to analyze the algorithm, given an input $(\mathcal{H}, X)$ of A^β, we define the measure

$$m(\mathcal{H}, X) = \begin{cases} \max\{|H| \mid H \in E(\mathcal{H}^{\bar{X}})\} & \text{, if } E(\mathcal{H}^{\bar{X}}) \neq \emptyset \\ 0 & \text{, otherwise.} \end{cases}$$

▶ **Lemma 23.** *The following statements hold:*

1. *If $A^\beta(\mathcal{H}, X)$ returns "yes", then* $\mathsf{mmhs}(\mathcal{H}) \geq \beta$.
2. *If there exists a minimal hitting set S^* of $\mathcal{H}$ such that $X \subseteq S^*$ and $|S^*| \geq \beta$, then $A^\beta(\mathcal{H}, X)$ returns "yes".*

The above properties imply that, given an instance $(\mathcal{H}, \beta)$ of MMHS, *$A^\beta(\mathcal{H}, \emptyset)$ returns "yes" if and only if* $\mathsf{mmhs}(\mathcal{H}) \geq \beta$.

Proof: We use the notation introduced in Definition 22. By Observation 1, recall that $\mathcal{H}$ has no hyperedge $H \in E(\mathcal{H})$ such that $H = \emptyset$. Moreover, observe that since $\mathcal{H}$ does not contain an empty hyperedge, by Observation 1, $m(\mathcal{H}, X) = 0$ is equivalent to $E(\mathcal{H}^{\bar{X}}) = \emptyset$.

We prove both properties by induction on $m(\mathcal{H}, X)$. Let us start with the first property.

Suppose that $A^\beta(\mathcal{H}, X)$ returns "yes". If $m(\mathcal{H}, X) = 0$, then $E(\mathcal{H}^{\bar{X}}) = \emptyset$, and the claimed property is true. Let is now assume that $m(\mathcal{H}, X) > 0$. As $E(\mathcal{H}^{\bar{X}}) \neq \emptyset$, the algorithm goes to the second case and chooses S. If $|S| \geq \beta$, then by Lemma 21 we get that $\mathsf{mmhs}(\mathcal{H}) \geq |S| \geq \beta$. Otherwise, there exists an is S_1 of $\mathcal{H}^{\bar{X}}$ such that $A^\beta(\mathcal{H}_{S_1}, X \cup (S \setminus S_1))$ returns "yes".

Let us argue that $m(\mathcal{H}_{S_1}, X \cup (S \setminus S_1)) < m(\mathcal{H}, X)$. Indeed, as S is a hitting set of $\mathcal{H}^{\bar{X}}$, each hyperedge in $E^{\bar{X}}$ intersects S. In particular, notice that $m(\mathcal{H}, X \cup S) = 0$ as $E^{\overline{X \cup S}} = \emptyset$. Recall that $\mathcal{H}_{S_1}$ is obtained from $\mathcal{H}$ by removing the vertices of S_1 from $V(\mathcal{H})$ and from the hyperedges containing such vertices. Note that $m(\mathcal{H}_{S_1}, X \cup (S \setminus S_1))$ corresponds to the maximum cardinality of a hyperedge of $E(\mathcal{H}_{S_1}^{\overline{X \cup (S \setminus S_1)}})$. By definition, such hyperedges correspond to hyperedges of $\mathcal{H}$ that do not intersect $X \cup (S \setminus S_1)$ in the hypergraph $\mathcal{H}_{S_1}$. Since S is a hitting set of $E(\mathcal{H}^{\bar{X}})$, the hyperedges in $E(\mathcal{H}_{S_1}^{\overline{X \cup (S \setminus S_1)}})$ necessarily contain at least one vertex of S_1. Since we removed the vertices in S_1 from all hyperedges they belong when constructing $\mathcal{H}_{S_1}$, each such hyperedge has lost at least one element. Thus, the inequality follows as for each hyperedge of $E(\mathcal{H}_{S_1}^{\overline{X\cup(S\setminus S_1)}})$ attaining $m(\mathcal{H}_{S_1}, X \cup (S \setminus S_1))$, there is a hyperedge in $E(\mathcal{H}^{\bar{X}})$ having at least one more vertex of S_1 as element.

As $m(\mathcal{H}_{S_1}, X \cup (S \setminus S_1)) < m(\mathcal{H}, X)$, by induction we get $\mathsf{mmhs}(\mathcal{H}_{S_1}) \geq \beta$, implying by Lemma 20 that $\mathsf{mmhs}(\mathcal{H}) \geq \beta$.

Let us now turn to the second property, and assume that there exists a minimal hitting set S^* of $\mathcal{H}$ such that $X \subseteq S^*$ and $|S^*| \geq \beta$. Suppose first that $m(\mathcal{H}, X) = 0$, implying that $E(\mathcal{H}^{\bar{X}}) = \emptyset$. In this case, we have that X is already a hitting set of $\mathcal{H}$, and thus, as S^* is minimal and $X \subseteq S^*$, we get that $S^* = X$, implying that $|X| \geq \beta$ and that the algorithm returns "yes". Suppose now that $m(\mathcal{H}, X) > 0$, implying $E(\mathcal{H}^{\bar{X}}) \neq \emptyset$, and thus that the algorithm goes to the second case and chooses S. If $|S| \geq \beta$ then we are done. Otherwise, since $|S| < \beta \leq |S^*|$, there is at least one vertex in $S^* \setminus S$. Let $S_1^* = S \setminus S^*$ and $S_2^* = S \cap S^*$. Since S^* is a hitting set of $\mathcal{H}$, there is no $H \in E(\mathcal{H})$ such that $H \subseteq S_1^*$. This implies that S_1^* is an is. Consequently, we have $S_1^* \in \mathcal{L}$. Thanks to Lemma 20, we have that S^* is also a minimal hitting set of $\mathcal{H}_{S_1^*}$. Since $X \cup (S \setminus S_1^*) \subseteq S^*$, and $m(\mathcal{H}_{S_1^*}, X \cup (S \setminus S_1^*)) < m(\mathcal{H}, X)$, we use the induction hypothesis to conclude that $A^\beta(\mathcal{H}_{S_1^*}, X \cup (S \setminus S_1^*))$ returns "yes". Therefore, $A(\mathcal{H}, X)$ returns "yes" as well. ◻

▶ **Theorem 24.** MMHS/$(\alpha + \beta)$ *can be solved in time* $\mathcal{O}^*(2^{\alpha\beta})$.

Proof: Given an instance $(\mathcal{H}, \beta)$ of MMHS, we simply call $A^\beta(\mathcal{H}, \emptyset)$. According to Lemma 23, this algorithm correctly decides whether $\mathsf{mmhs}(\mathcal{H}) \geq \beta$. Let us now analyze the running time. Let $f(\beta, \alpha, n)$ be the worst case running time of the algorithm $A^\beta(\mathcal{H}, X)$ when $m(\mathcal{H}, X) \leq \alpha$ and $|V(\mathcal{H})| = n$. We get that there exists a polynomial p_1 such that $f(\beta, 0, n) \leq p_1(n)$, as when $m(\mathcal{H}, X) = 0$ we have $E(\mathcal{H}^{\bar{X}}) = \emptyset$ and the algorithm only checks whether X is a minimal hitting set of size at least β. Otherwise, the algorithm first builds a minimal hitting set S of $\mathcal{H}^{\bar{X}}$. This can also be done in polynomial time, say $p_2(n)$. The worst case happens if $|S| \leq \beta - 1$. In this case, we have a recursive call for each possible subset $S_1 \subseteq S$ that is an is. We potentially have $2^{\beta-1}$ recursive calls, but we ensure that the cardinality of the largest hyperedge not hit decreases by at least one unit, i.e. we just need $f(\beta, \alpha - 1, n)$ steps for each recursive call in the worst case. Thus, in order to evaluate f, let $p(n) = p_1(n) + p_2(n)$ and $b = 2^{\beta-1}$. Observe that

$$f(\beta, \alpha, n) \leq \begin{cases} p(n) + bf(\beta, \alpha - 1, n) & \text{, if } \alpha \geq 1; \\ p(n) & \text{, otherwise.} \end{cases}$$

Thus, by simple substitution, note that

$$f(\beta, \alpha, n) \leq (b^{i-1} + \ldots + b) \cdot p(n) + b^i \cdot f(\beta, \alpha - i, n).$$

Consequently, by taking $i = \alpha$, we deduce that

$$f(\beta, \alpha, n) \leq p(n) \cdot \left(\frac{b^{\alpha+1} - 1}{b - 1}\right) = \left(\frac{(2^{\beta-1})^{\alpha+1} - 1}{2^{\beta-1} - 1}\right).$$

◻

4 MMBS parameterized by treewidth

In this section we prove that MMBS/tw is FPT. The algorithm requires a long case analysis. As one may expect, we present a dynamic programming (DP) algorithm using a nice tree decomposition of the input graph G. In Section 4.1, we present the notation we need and we provide some intuition about the parameters that we store in the tables of the algorithm. In Sections 4.2, 4.3, and 4.4 we present how to compute the table entries for a join, introduce, and forget node, respectively. In Section 4.5 we combine the previous ingredients to complete the algorithm.

4.1 Preliminaries

Consider a graph G and subsets $X \subseteq V(G)$, $B \subseteq V(G)$, and $Z \subseteq X$ such that Z is an is of G. We say that a set $I \subseteq V(G)$ is an (X, Z)-is if I is an is of G such that $I \cap X = Z$. We denote by

- $\alpha_{(X,Z)}(G)$ the size of a largest (X, Z)-is in G, and by
- $\alpha^B_{(X,Z)}(G)$ the size of a largest (X, Z)-is I in G such that $I \cap B = \emptyset$.

In both cases, if such a set does not exist, we set the corresponding parameter to $-\infty$. We say that a set $B \subseteq V(G)$ is an (X, Z)-bs in G if $\alpha^B_{(X,Z)}(G) < \alpha_{(X,Z)}(G)$, and we say that (X, Z) is *blocked* by B in G. These concepts are illustrated in Figure 2. Observe that if $B \cap Z \neq \emptyset$ then B is an (X, Z)-bs (as $\alpha^B_{(X,Z)}(G) = -\infty$), but the backward implication is not necessarily true as B may contain one vertex in $X \setminus Z$ of each maximum (X, Z)-is of G. Observe also that an $(\emptyset, \emptyset)$-is of G is simply an is of G, implying that $\alpha_{(\emptyset,\emptyset)}(G) = \alpha(G)$. Similarly, an $(\emptyset, \emptyset)$-bs is a bs of G.

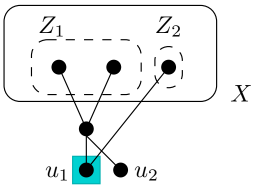

Figure 2 In this example there is only one maximum (X, Z_1)-is which is $I = Z_1 \cup \{u_1, u_2\}$. Note that $B = \{u_1\}$ is an (X, Z_1)-bs, but B is not an (X, Z_2)-bs.

In what follows we assume that we are given a nice tree decomposition $\mathcal{D} = (T, \mathcal{B})$ of the input graph G as defined in Section 2. In particular, recall that

- every node of T has at most two children,
- if a bag X corresponds to a node of T having two children with bags X^L and X^R, then $X = X^L = X^R$ (the node corresponding to X is a *join* node);
- if a bag X corresponds to a node of T having one child with bag X^C, then
  - either $X \subsetneq X^C$ and $|X^C| = |X| + 1$ (the node corresponding to X is a *forget* node), or
  - $X^C \subsetneq X$ and $|X| = |X^C| + 1$ (the node corresponding to X is an *introduce* node).

Discussion on the list of parameters used in the DP algorithm

As usual, our dynamic programming algorithm performs a leaf-to-root traversal of a nice tree decomposition $\mathcal{D} = (T, \mathcal{B})$ of an input graph G computing, for each node of the corresponding tree T, a set of tuples from the corresponding tuples of its children. Let us first explain the intuition behind each parameter of such tuples we shall compute and why they are needed. The formal details are presented in Definition 26 (page 24).

To simplify the presentation, we call a bag of the tree decomposition *join bag* (resp. *forget bag*, *introduce bag*) if its corresponding node is a join node (resp. forget node, introduce node). We also speak about the *children* of a bag, meaning the bags corresponding to the children of the considered node.

Consider a join bag X with children $X^L = X^R = X$, and suppose we look for a maximum mbs B of G_X. First, finding separately a maximum mbs B^L in G_{X^L} and B^R in G_{X^R} will not guarantee that the size of $B^L \cup B^R$ (assuming $B^L \cup B^R$ is an mbs in G_X) is maximum, and thus we introduce a parameter $B_0 \subseteq X$ and look for an mbs B of the graph G_X such that $B \cap X = B_0$.

Let I be a mis of G_X, $I^L = I \cap V(G_{X^L})$, and $I^R = I \cap V(G_{X^R})$. Observe that I^L (resp. I^R) is not necessarily a mis of G_{X^L} (resp. G_{X^R}), and thus it is pointless to find an mbs B^L (resp. B^R) in G_{X^L} (resp. G_{X^R}), as blocking maximum independent sets of G_{X^L} and G_{X^R} may not imply that we block maximum independent sets of G_X. This motivates the above notion of (X, Z)-is in G_X. More precisely, let $\mathcal{L} = \{Z \subseteq X \mid$ there exists a mis I of G_X such that $I \cap X = Z\}$. Then, B is an mbs of G_X if and only if:

1. (blocking condition) for every $Z \in \mathcal{L}$, B is an (X, Z)-bs in G_X, and
2. (minimality condition) for every $v \in B$, there must exist $Z \in \mathcal{L}$ such that $B \setminus \{v\}$ is not an (X, Z)-bs in G_X.

This explains why we have, in our list of parameters of our dynamic programming (and also the input of our auxiliary problem in Definition 27), a list $\mathcal{L}$ of subsets of X, in addition to the set B_0. Notice that there may exist $Z \in \mathcal{L}$ with $Z = \emptyset$. Toward the correct notion of the operator '$\vdash$' given in Definition 26, let us introduce some intermediate ones that we denote by '$\vdash_0$', '$\vdash_1$', '$\vdash_2$' and whose scope is limited to this preliminary discussion (as they will not be used in the eventual DP algorithm). Given $(X, B_0, \mathcal{L})$ and a set B, we say that $B \vdash_0 (X, B_0, \mathcal{L})$ if and only if

- $B \cap X = B_0$, and
- B satisfies the two properties above (blocking and minimality conditions).

Such a set B will be called a *solution* to $(X, B_0, \mathcal{L})$ (instead of bs).

Let us now argue that these three parameters are still not sufficient to design our algorithm, by exhibiting two situations where we suppose that we computed "small solutions", but extending these small solutions to the current bag creates a solution which no longer respects the minimality condition.

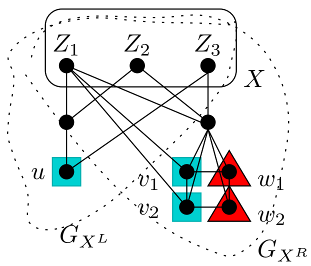

Figure 3 $B \vdash_0 (X, \emptyset, \mathcal{L})$ where $\mathcal{L} = \{Z_1, Z_2, Z_3\}$, $B = \{u, v_1, v_2\}$, and $B' = \{u, w_1, w_2\}$.

Let us start with the first situation. Suppose first that we have a solution $B \vdash_0 (X, B_0, \mathcal{L})$ for some $\mathcal{L} = \{Z_1, Z_2, Z_3\}$, as depicted in Figure 3. Recall that $X = X^L = X^R$ and let $B^L = B \cap V(G_{X^L})$ and $B^R = B \cap V(G_{X^R})$. We prove in Lemma 31 (page 25) that, for every $Z \in \mathcal{L}$, B is an (X, Z)-bs in G_X if and only if B^L is an (X^L, Z)-bs in G_{X^L} or B^R is an (X^R, Z)-bs in G_{X^R}. Thus, it may be the case, as in Figure 3, that $B^L = \{u\}$ is an (X^L, Z)-bs in G_{X^L} for $Z \in \mathcal{L}^L = \{Z_1, Z_2\}$, and $B^R = \{v_1, v_2\}$ is an (X^R, Z)-bs in

G_{X^R} for $Z \in \mathcal{L}^R = \{Z_2, Z_3\}$. Suppose now that we compute B'^L and B'^R such that $B'^L \vdash_0 (X^L, B_0, \mathcal{L}^L)$ and $B'^R \vdash_0 (X^R, B_0, \mathcal{L}^R)$, and let $B' = B'^L \cup B'^R$. It may be the case that B' does not verify the previous minimality condition 2. Indeed, let $u \in B'^L \setminus X$ and suppose that $B'^L \setminus \{u\}$ is not an (X^L, Z_1)-bs in G_{X^L}. Unfortunately, if B'^R is an (X^R, Z_1)-bs in G_{X^R} (even if $Z_1 \notin \mathcal{L}^R$), we will have that $B' \setminus \{u\}$ is still a (X, Z_1)-bs in G_X, and thus maybe still an (X, Z)-bs for any $Z \in \mathcal{L}$. We overcome this problem by forcing B'^R *not* to be an (X^R, Z_1)-bs in G_{X^R}. This explains why we have in the input a list $\mathcal{S}$ of subsets of X, and we now impose that for any $Z \in \mathcal{S}$, B must not be an (X, Z)-bs in G_X.

Thus, now we denote by $B \vdash_1 (X, B_0, \mathcal{L}, \mathcal{S})$ the property that $B \vdash_0 (X, B_0, \mathcal{L})$ and, for any $Z \in \mathcal{S}$, B is not an (X, Z)-bs in G_X.

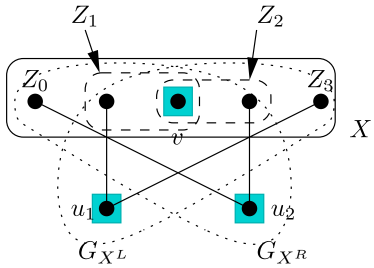

Figure 4 $B_0 = \{v\}$, $B'^L = \{u_1, v\}$, and $B'^R = \{u_2, v\}$.

Let us now turn to the second situation, which is depicted Figure 4, where $\mathcal{L} = \{Z_0, Z_1, Z_2, Z_3\}$ and $B_0 = \{v\}$. Suppose that we compute B'^L and B'^R such that $B'^L \vdash_1 (X^L, B_0, \mathcal{L}^L, \mathcal{S}^L)$ where $\mathcal{L}^L = \{Z_0, Z_1, Z_2\}$, $\mathcal{S}^L = \{Z_3\}$ and $B'^R \vdash_1 (X^R, B_0, \mathcal{L}^R, \mathcal{S}^R)$ where $\mathcal{L}^R = \{Z_1, Z_2, Z_3\}$ and $\mathcal{S}^R = \{Z_0\}$. Let $B' = B'^L \cup B'^R$. Let $v \in B_0$. By minimality condition 2, there exists $Z \in \mathcal{L}^L$ such that $B'^L \setminus \{v\}$ is not an (X^L, Z)-bs in G_{X^L} (where $Z = Z_1$ in Figure 4). In the same way, there exists $Z' \in \mathcal{L}^R$ such that $B'^R \setminus \{v\}$ is not an (X^R, Z')-bs in G_{X^R} (where $Z' = Z_2$ in Figure 4). If $Z \neq Z'$, we may not be able to conclude that there exists a $Z'' \in \mathcal{L}$ such that $B' \setminus \{v\}$ is not an (X, Z'')-bs in G_X. In the example depicted in Figure 4, $B' \setminus \{v\}$ is still an (X, Z)-bs in G_X for every $Z \in \mathcal{L}$. Thus, for $v \in B_0$, we will keep control of the minimality condition in a more precise way by

- introducing another list $\mathcal{L}_2$ of subsets of X, and still ask that B is an (X, Z)-bs in G_X for any $Z \in \mathcal{L}_2$,
- introducing a function $f : B_0 \to \mathcal{L}_2$, and
- (minimality condition in B_0) requiring that for every $v \in B_0$, $B \setminus \{v\}$ is not an $(X, f(v))$-bs in G_X.

In the previous example, we would have to set either $f^L(v) = f^R(v) = Z_1$ or $f^L(v) = f^R(v) = Z_2$, but none of these choices leads to a feasible solution on both the left and the right hand sides. This is not surprising, as in fact there is no set B such that $B \vdash_1 (X, B_0, \mathcal{L}, \mathcal{S})$ where $\mathcal{L} = \{Z_0, Z_1, Z_2, Z_3\}$, $B_0 = \{v\}$, and $\mathcal{S} = \emptyset$. Indeed, being an (X, Z_0)-bs in G_X forces any B to contain u_1 (as B cannot contain the vertex of Z_0), and being a (X, Z_3)-bs in G_X forces any B to contain u_2. This means that we necessarily have $\{u_1, u_2, v\} \subseteq B$. Then, observe that B is not minimal as $\{u_1, u_2\}$ is still a (X, Z)-bs in G_X for any $Z \in \mathcal{L}$. The conclusion is that in this situation, forcing v to be in any solution leads to an infeasible instance.

Finally, even when using function f, and defining accordingly $B \vdash_2 (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$ if $B \vdash_1 (X, B_0, \mathcal{L}_1, \mathcal{S})$ and B respects the previous minimality condition in B_0, there is a last important detail. Suppose $B'^L \vdash_2 (X^L, B_0, \mathcal{L}_1^L, \mathcal{L}_2^L, f^L, \mathcal{S}^L)$ where for example that

$\mathcal{L}_1^L = \{Z_1\}$, $\mathcal{L}_2^L = \{Z_2\}$, and consider $v \in B'^L \setminus B_0$. We know that there exists Z such that $B'^L \setminus \{v\}$ is not an (X^L, Z)-bs in G_{X^L}, but we must even impose that $Z \in \mathcal{L}_1^L$, as otherwise if $Z = Z_2$ then $B' \setminus \{v\}$ would still be an (X, Z)-bs in G_X. Thus, the minimality condition is finally as follows:

1. (minimality condition outside B_0, forcing $Z \in \mathcal{L}_1$) $\forall v \in B \setminus B_0$, $\exists Z \in \mathcal{L}_1$ such that $B \setminus \{v\}$ is not an (X, Z)-bs in G_X.
2. (minimality condition in B_0) $\forall v \in B_0$, $B \setminus \{v\}$ is not an $(X, f(v))$-bs in G_X, where $f(v) \in \mathcal{L}_2$.

Even if we only discussed here the case where X is a join node, it turns out that this list of parameters is also enough for the introduce and forget nodes.

Defining the auxiliary problem

Let us now define the auxiliary problem that will be solved by our DP algorithm.

▶ **Definition 25.** *Let G be a graph and let $\mathcal{D} = (T, \mathcal{B})$ be a nice tree decomposition of G. Let $\mathcal{E}(G, \mathcal{D})$ be the set containing all tuples $(X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$ such that:*

- $X \in \mathcal{B}$*,*
- $B_0 \subseteq X$*,*
- $\mathcal{L}_1, \mathcal{L}_2, \mathcal{S} \subseteq 2^X$ *such that for every $Z \in \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{S}$, Z is an* is *of G, and*
- $f : B_0 \to \mathcal{L}_2$*.*

▶ **Definition 26.** *Let G be a graph and let $\mathcal{D} = (T, \mathcal{B})$ be a nice tree decomposition of G. For every $(X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S}) \in \mathcal{E}(G, \mathcal{D})$ and $B \subseteq V(G_X)$, we write $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$ if and only if*

i) $B \cap X = B_0$*,*
ii) $\forall Z \in \mathcal{L}_1 \cup \mathcal{L}_2$*, B is an (X, Z)-*bs *in G_X,*
iii) $\forall Z \in \mathcal{S}$*, B is not an (X, Z)-*bs *in G_X,*
iv) *and the following two minimality conditions are satisfied:*
  a) $\forall v \in B \setminus B_0$*, $\exists Z \in \mathcal{L}_1$ such that $B \setminus \{v\}$ is not an (X, Z)-*bs *in G_X, and*
  b) $\forall v \in B_0$*, $B \setminus \{v\}$ is not an $(X, f(v))$-*bs *in G_X.*

Let us point out that there may exist $Z \in \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{S}$ with $Z = \emptyset$, and that if $\exists Z \in \mathcal{S}$ such that $B_0 \cap Z \neq \emptyset$, then there is no solution (because of Property iii).

▶ **Definition 27.** *We define the optimization problem Π as follows, where we consider that the input graph G and a nice tree decomposition $\mathcal{D}$ of G are fixed:*

Input: A tuple $(X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S}) \in \mathcal{E}(G, \mathcal{D})$.
Output: A set $B \subseteq V(G_X)$ such that $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$.
Objective: Maximize $|B|$.

We say that an instance I of Π is *feasible* if there exists a set B such that $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$. Let us now show that being able to solve optimally problem Π is sufficient for computing the parameter mmbs(G), for a given graph G.

▶ **Proposition 28.** *Let G be a graph and $\mathcal{D} = (T, \mathcal{B})$ be a nice tree decomposition of G such that T is rooted at $X_0 = \{\emptyset\}$. For every $B \subseteq V(G)$,*

$$B \vdash (\emptyset, \emptyset, \{\emptyset\}, \emptyset, \emptyset, \emptyset) \text{ if and only if } B \text{ is an mbs of } G.$$

Proof: Let $(X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S}) = (\emptyset, \emptyset, \{\emptyset\}, \emptyset, \emptyset, \emptyset)$. Recall that being an $(\emptyset, \emptyset)$-bs in G_{X_0} is equivalent to being a bs in G.

Suppose first that $B \vdash (\emptyset, \emptyset, \{\emptyset\}, \emptyset, \emptyset, \emptyset)$. By Property ii, B is an $(\emptyset, \emptyset)$-bs in G_{X_0}, implying that B is a bs of G. Let us now prove that B is minimal. Let $v \in B$. As $v \in B \setminus B_0$, by Property iva, there exists $Z \in \mathcal{L}_1$ such that $B \setminus \{v\}$ is not an (X_0, Z)-bs in G. As $\mathcal{L}_1 = \{\emptyset\}$, we obtain that $B \setminus \{v\}$ is not an $(\emptyset, \emptyset)$-bs in G, and thus not a bs in G.

Suppose now that B is an mbs of G. Property ii is satisfied as B is a bs in G. Let us now prove Property iva. Let $v \in B \setminus B_0$. As B is minimal, $B \setminus \{v\}$ is not a bs in G, and thus not an $(\emptyset, \emptyset)$-bs in $G_{X_0} = G$, where $\emptyset \in \mathcal{L}_1$. □

The following proposition is now immediate.

▶ **Proposition 29.** *Given an n-vertex graph G with treewidth $\mathsf{tw}(G) = t$, if*

- *$t_1(n, t)$ is the time to compute a nice tree decomposition $\mathcal{D}$ of G of width t, and*
- *$t_A(n, t)$ is the time to compute an optimal solution of problem Π,*

then one can compute $\mathsf{mmbs}(G)$ in time $\mathcal{O}(t_1(n, t) + t_A(n, t))$.

In what follows, namely in Sections 4.2, 4.3, and 4.4, we fix an input graph G and a nice tree decomposition $\mathcal{D} = (\mathcal{B}, T)$ of G of width t.

4.2 Join node

Before proving Lemma 32 corresponding to the join case, let us first prove the following technical lemmas.

▶ **Lemma 30.** *For every $I \subseteq V(G_X)$ and every $Z \subseteq X$ where Z is an is of G_X, I is a maximum (X, Z)-is in G_X if and only if $I^L = I \cap V(G_{X^L})$ is a maximum (X^L, Z)-is in G_{X^L} and $I^R = I \cap V(G_{X^R})$ is a maximum (X^R, Z)-is in G_{X^R}.*

Proof: For the forward implication, suppose I is a maximum (X, Z)-is in G_X. Let I^L_* be a maximum (X^L, Z)-is in G_{X^L}. Then, we claim that $(I \setminus I^L) \cup I^L_*$ is still an is. Indeed, notice that there is no edge between $I^L_* \cap X$ and I^R, because $I^L_* \cap X = I^L \cap X = Z$. Moreover, there is also no edge between $I^L_* \setminus X$ and $I^R \setminus X$ as, by the properties of a tree decomposition, there is no edge even between $V(G_{X_L}) \setminus X$ and $V(G_{X_R}) \setminus X$.

As $((I \setminus I^L) \cup I^L_*) \cap X = Z$, this implies that $(I \setminus I^L) \cup I^L_*$ is an (X, Z)-is in G_X and that $|I| \geq (I \setminus I^L) \cup I^L_*$. As $I \cap I^L = I \cap I^L_* = Z$, we get $|I^L| \geq |I^L_*|$. As $I^L \cap X^L = Z$, we obtain that I^L is a maximum (X^L, Z)-is in G_{X^L}. The same arguments hold for I^R.

For the backward implication, suppose I^L is a maximum (X^L, Z)-is in G_{X^L} and I^R is a maximum (X^R, Z)-is in G_{X^R}. Observe first that $I^L \cup I^R$ is an (X, Z)-is in G_X, as there is no edge between $V(G_{X_L}) \setminus X$ and $V(G_{X_R}) \setminus X$. Let I^* be a maximum (X, Z)-is in G_X. Note that $I^L_* := I^* \cap V(G_{X^L})$ is an (X^L, Z)-is in G_{X^L}, and symmetrically that $I^R_* := I^* \cap V(G_{X^R})$ is an (X^R, Z)-is in G_{X^R}. This implies that $|I^L_*| \leq |I^L|$ and $|I^R_*| \leq |I^R|$. As $I^L_* \cap I^R_* = I^L \cap I^R = Z$, the previous inequalities imply $|I^*| \leq |I|$, meaning that I is a maximum (X, Z)-is in G_X. □

▶ **Lemma 31.** *Let $Z \subseteq X$. For every $B \subseteq V(G_X)$, B is an (X, Z)-bs in G_X if and only if $B^L = B \cap V(G_{X^L})$ is an (X^L, Z)-bs in G_{X^L} or $B^R = B \cap V(G_{X^R})$ is an (X^R, Z)-bs in G_{X^R}.*

Proof: For the forward implication, suppose B is an (X, Z)-bs in G_X. Suppose by contradiction that there exists a maximum (X^L, Z)-is I^L in G_{X^L} such that $I^L \cap B^L = \emptyset$, and a maximum (X^R, Z)-is I^R in G_{X^R} such that $I^R \cap B^R = \emptyset$. Let $I = I^L \cup I^R$. By Lemma 30, I

is a maximum (X,Z)-is in G_X. As $I^L \cap B = I^L \cap B^L = \emptyset$, and also $I^R \cap B = I^R \cap B^R = \emptyset$, we get $I \cap (B^L \cup B^R) = I \cap B = \emptyset$, a contradiction to the hypothesis that B is an (X,Z)-bs in G_X.

For the backward implication, suppose B^L is an (X^L,Z)-bs in G_{X^L} or B^R is an (X^R,Z)-bs in G_{X^R}. Suppose by contradiction that there exists a maximum (X,Z)-is I in G_X such that $I \cap B = \emptyset$. By Lemma 30, $I^L = I \cap V(G_{X^L})$ is a maximum (X^L,Z)-is in G_{X^L} and $I^R = I \cap V(G_{X^R})$ is a maximum (X^R,Z)-is in G_{X^R}. As $I^L \cap B = I^R \cap B = \emptyset$, we obtain that B^L is not an (X^L,Z)-bs in G_{X^L} and that B^R is not an (X^R,Z)-bs in G_{X^R}, a contradiction. ◻

We are now ready to state the main lemma of this section.

▶ **Lemma 32.** *Let $(X,B_0,\mathcal{L}_1,\mathcal{L}_2,f,\mathcal{S}) \in \mathcal{E}(G,\mathcal{D})$ where $X \in \mathcal{B}$ is a join node and X^L, X^R are the children of X (with $X = X^L = X^R$). For every $B \subseteq V(G_X)$, it holds that $B \vdash (X,B_0,\mathcal{L}_1,\mathcal{L}_2,f,\mathcal{S})$ if and only if there exist sets $B^L, B^R, \mathcal{L}_1^A, \mathcal{L}_1^B, \mathcal{L}_1^C, \mathcal{L}_2^A, \mathcal{L}_2^B, \mathcal{L}_2^C$ such that the following properties hold:*

1. $B = B^L \cup B^R$,
2. $\mathcal{L}_1 = \mathcal{L}_1^A \uplus \mathcal{L}_1^B \uplus \mathcal{L}_1^C$ *and* $\mathcal{L}_2 = \mathcal{L}_2^A \uplus \mathcal{L}_2^B \uplus \mathcal{L}_2^C$,
3. *for every* $v \in B_0$, $f(v) \in \mathcal{L}_2^B$,
4. $B^L \vdash (X^L, B_0, \mathcal{L}_1^L, \mathcal{L}_2^L, f^L, \mathcal{S}^L)$, *where*
   - $\mathcal{L}_1^L = \mathcal{L}_1^A$,
   - $\mathcal{L}_2^L = \mathcal{L}_1^B \cup \mathcal{L}_2^B \cup \mathcal{L}_2^A$,
   - $f^L = f$, *and*
   - $\mathcal{S}^L = \mathcal{S} \cup \mathcal{L}_1^C \cup \mathcal{L}_2^C$;

   and $B^R \vdash (X^R, B_0, \mathcal{L}_1^R, \mathcal{L}_2^R, f^R, \mathcal{S}^R)$, *where*

   - $\mathcal{L}_1^R = \mathcal{L}_1^C$,
   - $\mathcal{L}_2^R = \mathcal{L}_1^B \cup \mathcal{L}_2^B \cup \mathcal{L}_2^C$,
   - $f^R = f$, *and*
   - $\mathcal{S}^R = \mathcal{S} \cup \mathcal{L}_1^A \cup \mathcal{L}_2^A$.

Proof: For the forward implication, suppose first that $B \vdash (X,B_0,\mathcal{L}_1,\mathcal{L}_2,f,\mathcal{S})$. Let $B^L = B \cap V(G_{X^L})$ and $B^R = B \cap V(G_{X^R})$, satisfying Property 1 of the lemma. For $i \in [2]$, let

- $\mathcal{L}_i^A = \{Z \in \mathcal{L}_i \mid B^L$ is an (X^L,Z)-bs in G_{X^L} and B^R is not an (X^R,Z)-bs in $G_{X^R}\}$,
- $\mathcal{L}_i^C = \{Z \in \mathcal{L}_i \mid B^L$ is not an (X^L,Z)-bs in G_{X^R} and B^R is an (X^R,Z)-bs in $G_{X^R}\}$, and
- $\mathcal{L}_i^B = \{Z \in \mathcal{L}_i \mid B^L$ is an (X^L,Z)-bs in G_{X^L} and B^R is an (X^R,Z)-bs in $G_{X^R}\}$.

By Definition 26, for every $Z \in \mathcal{L}_i$, as B is an (X,Z)-bs in G_X. By Lemma 31, we obtain that B^L is an (X^L,Z)-bs in G_{X^L} or B^R is an (X^R,Z)-bs in G_{X^R}. This implies that $\mathcal{L}_i = \mathcal{L}_i^A \uplus \mathcal{L}_i^B \uplus \mathcal{L}_i^C$, and thus Property 2 is satisfied.

For Property 4, let us only prove that $B^L \vdash (X^L, B_0, \mathcal{L}_1^L, \mathcal{L}_2^L, f^L, \mathcal{S}^L)$, as the proof for B^R follows the same arguments. We verify that each of the (non-trivial) properties of Definition 26 is satisfied.

Property ii. We need to prove that B^L is an (X^L,Z)-bs in G_{X^L} for every $Z \in \mathcal{L}_1^L \cup \mathcal{L}_2^L$. This follows from the definition of the sets $\mathcal{L}_i^L$, and by the hypothesis that B is an (X,Z)-bs in G_X for every $Z \in \mathcal{L}_1 \cup \mathcal{L}_2$ (since $B \vdash (X,B_0,\mathcal{L}_1,\mathcal{L}_2,f,\mathcal{S})$).

Property iii. Let us prove that B^L is not an (X^L,Z)-bs in G_{X^L}, for every $Z \in \mathcal{S}^L = \mathcal{S} \cup \mathcal{L}_1^C \cup \mathcal{L}_2^C$. Let $Z \in \mathcal{S}^L$. If $Z \in \mathcal{S}$, then since B is not an (X,Z)-bs in G_X, because

$B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, we have by Lemma 31 that B^L is not an (X^L, Z)-bs in G_{X^L}. If $Z \in \mathcal{L}_1^C \cup \mathcal{L}_2^C$, then the result follows from definition of $\mathcal{L}_i^C$.

Property iva. We have to prove that $\forall v \in B^L \setminus B_0$, $\exists Z \in \mathcal{L}_1^L = \mathcal{L}_1^A$ such that $B^L \setminus \{v\}$ is not an (X^L, Z)-bs in G_{X^L}. If $B^L = B_0$, the statement trivially holds. Otherwise, let $v \in B^L \setminus B_0$. As $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, there exists $Z \in \mathcal{L}_1$ such that $B \setminus \{v\}$ is not an (X, Z)-bs in G_X, by Definition 26. This implies, by Lemma 31, that $B^L \setminus \{v\}$ is not an (X^L, Z)-bs in G_{X^L}. As $B \setminus \{v\} \supseteq B^R$, we get that B^R is not an (X, Z)-bs in G_X, and thus by Lemma 31 that B^R is not an (X^R, Z)-bs in G_{X^R}, implying that $Z \in \mathcal{L}_1^A$.

Property ivb. We finally have to prove that $\forall v \in B_0$, $B^L \setminus \{v\}$ is not an $(X_L, f^L(v))$-bs in G_{X^L}. Let $v \in B_0$. Let $Z = f^L(v) = f(v)$, where $Z \in \mathcal{L}_2$. As $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, $B \setminus \{v\}$ is not an (X, Z)-bs in G_X, implying by Lemma 31 that $B^L \setminus \{v\}$ is not an (X^L, Z)-bs in G_{X^L}. Moreover, as B is an (X, Z)-bs in G_X and $B \setminus \{v\}$ is not an (X, Z)-bs in G_X, we deduce that $v \in Z$, implying that $Z \in \mathcal{L}_2^B \subseteq \mathcal{L}_2^L$. Note also that $f^L(v) \in \mathcal{L}_2^B$, as required by Property 3.

For the backward implication, suppose that there exist $B^L, B^R, \mathcal{L}_1^A, \mathcal{L}_1^B, \mathcal{L}_1^C, \mathcal{L}_2^A, \mathcal{L}_2^B, \mathcal{L}_2^C$ satisfying the lemma's conditions. Let us prove that $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, by verifying again that each of the (non-trivial) properties of Definition 26 is satisfied.

Property ii. We have to prove that $B = B^L \cup B^R$ is an (X, Z)-bs in G_X, for every $Z \in \mathcal{L}_1 \cup \mathcal{L}_2$. By hypothesis, we know that: $B^L \vdash (X^L, B_0, \mathcal{L}_1^L, \mathcal{L}_2^L, f^L, \mathcal{S}^L)$ and $B^R \vdash (X^R, B_0, \mathcal{L}_1^R, \mathcal{L}_2^R, f^R, \mathcal{S}^R)$. By Definition 26, we deduce that B^L is an (X^L, Z)-bs in G_{X^L}, for every $Z \in \mathcal{L}_1^L \cup \mathcal{L}_2^L = \mathcal{L}_1^A \cup \mathcal{L}_2^A \cup \mathcal{L}_1^B \cup \mathcal{L}_2^B$. By Lemma 31, B is an (X, Z)-bs in G_X, for every $Z \in \mathcal{L}_1^A \cup \mathcal{L}_2^A \cup \mathcal{L}_1^B \cup \mathcal{L}_2^B$. Analogously, one may deduce that B is an (X, Z)-bs in G_X, for every $Z \in \mathcal{L}_1^C \cup \mathcal{L}_2^C \cup \mathcal{L}_1^B \cup \mathcal{L}_2^B$. Thus, B is an (X, Z)-bs in G_X, for every $Z \in \mathcal{L}_1 \cup \mathcal{L}_2$.

Property iii. We have to prove that B is not an (X, Z)-bs in G_X, for every $Z \in \mathcal{S}$. Let $Z \in \mathcal{S}$. Since $B^L \vdash (X^L, B_0, \mathcal{L}_1^L, \mathcal{L}_2^L, f^L, \mathcal{S}^L)$ and $\mathcal{S} \subseteq \mathcal{S}^L$, we have that B^L is not an (X^L, Z)-bs in G_{X^L}, by Definition 26. As B^L is not an (X^L, Z)-bs in G_{X^L} and, analogously, B^R is not an (X^R, Z)-bs in G_{X^R}, it implies by Lemma 31 that B is not an (X, Z)-bs in G_X.

Property iva. Let us now prove that for every $v \in B \setminus B_0$, there is $Z \in \mathcal{L}_1 = \mathcal{L}_1^A \uplus \mathcal{L}_1^B \uplus \mathcal{L}_1^C$ such that $B \setminus \{v\}$ is not an (X, Z)-bs in G_X. If $B = B_0$, then there is nothing to prove. Otherwise, let $v \in B \setminus B_0$, and suppose without loss of generality that $v \in B^L \setminus B_0$. As $B^L \vdash (X^L, B_0, \mathcal{L}_1^L, \mathcal{L}_2^L, f^L, \mathcal{S}^L)$, and as $\mathcal{L}_1^L = \mathcal{L}_1^A$, there exists $Z \in \mathcal{L}_1^A$ such that $B^L \setminus \{v\}$ is not an (X^L, Z)-bs in G_{X^L}. As $Z \in \mathcal{L}_1^A$, $L_1^A \subseteq \mathcal{S}^R$, and $B^R \vdash (X^R, B_0, \mathcal{L}_1^R, \mathcal{L}_2^R, f^R, \mathcal{S}^R)$, B^R is not an (X^R, Z)-bs in G_{X^R}. Thus, by Lemma 31, $B \setminus \{v\}$ is not an (X, Z)-bs in G_X.

Property ivb. Let us finally prove that for each $v \in B_0$, $B \setminus \{v\}$ is not an $(X, f(v))$-bs in G_X. Let $v \in B_0$ and let $Z = f(v)$. By Property 3, we know that $Z \in \mathcal{L}_2^B$, i.e. Z is both in $\mathcal{L}_2^L$ and $\mathcal{L}_2^R$. Then, as $B^L \vdash (X^L, B_0, \mathcal{L}_1^L, \mathcal{L}_2^L, f^L, \mathcal{S}^L)$ and $f^L = f$, by Property ivb we get that $B^L \setminus \{v\}$ is not an (X^L, Z)-bs in G_{X^L}. Using the same arguments for B^R, we get that $B^R \setminus \{v\}$ is not an (X^R, Z)-bs in G_{X^R}. By Lemma 31, we obtain that $B \setminus \{v\}$ is not an (X, Z)-bs in G_X. ◻

4.3 Introduce node

▶ **Definition 33.** *Let G be a graph, $X \subseteq V(G)$, $v \in X$, and $\mathcal{R} \subseteq 2^X$. We denote*

- $\mathcal{R}(v) = \{Z \in \mathcal{R} \mid v \in Z\}$,
- $\mathcal{R}(\bar{v}) = \{Z \in \mathcal{R} \mid v \notin Z\}$, *and*
- $r_v(\mathcal{R}) = \{Z \setminus \{v\} \mid Z \in \mathcal{R}\}$.

Before proving Lemma 38 corresponding to the introduce case, let us first prove the following lemmas where we assume that $X \in \mathcal{B}$ is an introduce node and that X^C is the child of X with $X^C = X \setminus \{v\}$ for some vertex $v \in X$.

▶ **Lemma 34.** *Let $Z \subseteq X$ such that Z is an* is *with $v \in Z$. For every $I \subseteq V(G)$ such that $v \in I$, I is a maximum (X, Z)-*is *in G_X if and only if $I \setminus \{v\}$ is a maximum $(X^C, Z \setminus \{v\})$-*is *in G_{X^C}.*

Proof: For the forward implication, suppose that I is a maximum (X, Z)-is in G_X such that $v \in I$. Note that $I \setminus \{v\}$ is an $(X^C, Z \setminus \{v\})$-is in G_{X^C}. Let I' be a maximum $(X^C, Z \setminus \{v\})$-is in G_{X^C}. Since $N_{G_X}(v) \subseteq X$ by the properties of a tree decomposition, and since Z is an is of G, we deduce that $I' \cup \{v\}$ is an (X, Z)-is in G_X, implying $|I' \cup \{v\}| \leq |I|$. Therefore $|I \setminus \{v\}| \geq |I'|$, hence $I \setminus \{v\}$ is a maximum $(X^C, Z \setminus \{v\})$-is in G_{X^C}.

For the backward implication, suppose that $I \setminus \{v\}$ is a maximum $(X^C, Z \setminus \{v\})$-is in G_{X^C}. Note that I is an (X, Z)-is in G_X. Let I' be a maximum (X, Z)-is in G_X. As $I' \setminus \{v\}$ is an $(X^C, Z \setminus \{v\})$-is in G_{X^C}, we get $|I' \setminus \{v\}| \leq |I \setminus \{v\}|$ and therefore, since both I and I' contain v, $|I'| \leq |I|$ and the lemma follows. ◻

▶ **Lemma 35.** *Let $Z \subseteq X$ such that Z is an* is *with $v \notin Z$. For every $I \subseteq V(G)$ such that $v \notin I$, I is a maximum (X, Z)-*is *in G_X if and only if I is a maximum (X^C, Z)-*is *in G_{X^C}.*

Proof: For the forward implication, suppose that I is a maximum (X, Z)-is in G_X such that $v \notin I$. Note that I is an (X^C, Z)-is in G_{X^C}. Let I' be a maximum (X^C, Z)-is in G_{X^C}. As I' is an (X, Z)-is in G_X, $|I'| \leq |I|$, leading to the desired result.

For the backward implication, consider that I is a maximum (X^C, Z)-is in G_{X^C}. Note that I is an (X, Z)-is in G_X. Let I' be a maximum (X, Z)-is in G_X. Since $v \notin Z$, $v \notin I'$ and I' is an (X^C, Z)-is in G_{X^C}, we deduce that $|I'| \leq |I|$. ◻

▶ **Lemma 36.** *Let $Z \subseteq X$ such that Z is an* is*. For every $B \subseteq V(G_X)$ such that $v \notin B$, B is an (X, Z)-*bs *in G_X if and only if B is an $(X^C, Z \setminus \{v\})$-*bs *in G_{X^C}.*

Proof: For the forward implication, assume that B is an (X, Z)-bs in G_X such that $v \notin B$. Let I be a maximum $(X^C, Z \setminus \{v\})$-is in G_{X^C}. Suppose first that $v \in Z$. By Lemma 34, we get that $I \cup \{v\}$ is a maximum (X, Z)-is in G_X, implying that $B \cap (I \cup \{v\}) \neq \emptyset$. As $v \notin B$, we get $B \cap I \neq \emptyset$. Suppose now that $v \notin Z$. By Lemma 35, we get that I is a maximum (X, Z)-is in G_X, implying $B \cap I \neq \emptyset$.

For the backward implication, suppose that B is an $(X^C, Z \setminus \{v\})$-bs in G_{X^C}. Let I be a maximum (X, Z)-is in G_X. Suppose first that $v \in Z$. By Lemma 34, we get that $I \setminus \{v\}$ is a maximum $(X^C, Z \setminus \{v\})$-is in G_{X^C}, implying that $B \cap (I \setminus \{v\}) \neq \emptyset$. Suppose now that $v \notin Z$. By Lemma 35, we get that I is a maximum (X^C, Z)-is in G_{X^C}, implying $B \cap I \neq \emptyset$.
◻

▶ **Lemma 37.** *Let $Z \subseteq X$ such that Z is an* is *with $v \notin Z$. For every $B \subseteq V(G_X)$ such that $v \in B$, B is an (X, Z)-*bs *in G_X if and only if $B \setminus \{v\}$ is an (X^C, Z)-*bs *in G_{X^C}.*

Proof: For the forward implication, suppose that B is an (X, Z)-bs in G_X such that $v \in B$. Let I be a maximum (X^C, Z)-is in G_{X^C}. By Lemma 35, I is a maximum (X, Z)-is in G_X, implying that $B \cap I \neq \emptyset$. As $v \notin I$, we get $(B \setminus \{v\}) \cap I \neq \emptyset$.

For the backward implication, suppose that $B \setminus \{v\}$ is an (X^C, Z)-bs in G_{X^C}. Let I be a maximum (X, Z)-is in G_X. By Lemma 35, I is a maximum (X^C, Z)-is in G_{X^C}, implying that $(B \setminus \{v\}) \cap I \neq \emptyset$. ◻

We are now ready to state the main lemma of this section. Let us recall that given a function $f : A \to B$ and a subset $A' \subseteq A$, we denote by $f_{|A'}$ the restriction of f to A'.

▶ **Lemma 38.** *Let $(X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S}) \in \mathcal{E}(G, \mathcal{D})$ where $X \in \mathcal{B}$ is an introduce node and X^C is the child of X with $X^C = X \setminus \{v\}$. For every $B \subseteq V(G_X)$, $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$ if and only if one of the following two cases holds:*

Case 1: *$v \in B$ and there exist $\mathcal{L}_2^A, \mathcal{L}_2^B$ such that*
1. *$\mathcal{L}_2(v) = \mathcal{L}_2^A \uplus \mathcal{L}_2^B$,*
2. *$f(v) \in \mathcal{L}_2^A$,*
3. *for every $Z \in \mathcal{S}$, $v \notin Z$, and*
4. *$B \setminus \{v\} \vdash (X \setminus \{v\}, B_0 \setminus \{v\}, \mathcal{L}_1^C, \mathcal{L}_2^C, f^C, \mathcal{S}^C)$, where*
   - *$\mathcal{L}_1^C = \mathcal{L}_1(\bar{v})$,*
   - *$\mathcal{L}_2^C = \mathcal{L}_2(\bar{v})$,*
   - *$f^C = f_{|B\setminus\{v\}}$, and*
   - *$\mathcal{S}^C = \mathcal{S} \cup r_v(\mathcal{L}_2^A)$.*

Case 2: *$v \notin B$ and $B \vdash (X \setminus \{v\}, B_0, \mathcal{L}_1^C, \mathcal{L}_2^C, f^C, \mathcal{S}^C)$, where*
- *$\mathcal{L}_1^C = r_v(\mathcal{L}_1)$,*
- *$\mathcal{L}_2^C = r_v(\mathcal{L}_2)$,*
- *$f^C(v') = f(v') \setminus \{v\}$ for every $v' \in B_0$, and*
- *$\mathcal{S}^C = r_v(\mathcal{S})$.*

Proof: For the forward implication, suppose that $B \subseteq V(G_X)$ is such that $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$. We distinguish the two cases considered in Lemma 38. In both cases, we verify that each of the corresponding properties is satisfied.

Case 1. Suppose that $v \in B$, and thus $v \in B_0$, as $v \in X$ and $B_0 = B \cap X$. Let $\mathcal{L}_2^A = \{Z \in \mathcal{L}_2 | \ B \setminus \{v\}$ is not an (X, Z)-bs in $G_X\}$. By Property ivb applied to v, we get that $f(v) \in \mathcal{L}_2^A$, implying Property 2. Moreover, for every $Z \in \mathcal{L}_2^A$, there exists a maximum (X, Z)-is I in G_X such that $I \cap (B \setminus \{v\}) = \emptyset$, and thus if we had $v \notin Z$, then $v \notin I$ and $I \cap B = \emptyset$, a contradiction. This implies that $\mathcal{L}_2^A \subseteq \mathcal{L}_2(v)$, and we define $\mathcal{L}_2^B = \mathcal{L}_2(v) \setminus \mathcal{L}_2^A$, implying Property 1. By Property iii, as B is not an (X, Z)-bs in G_X for every $Z \in \mathcal{S}$ and $v \in B$, we get Property 3. Let us now prove Property 4, by verifying each of the non-trivial properties of Definition 26 applied to $B \setminus \{v\}$.

Property ii. Recall that $X^C = X \setminus \{v\}$. We need to prove that $B \setminus \{v\}$ is an (X^C, Z)-bs in G_{X^C}, for every $Z \in \mathcal{L}_1^C \cup \mathcal{L}_2^C = \mathcal{L}_1(\bar{v}) \cup \mathcal{L}_2(\bar{v})$. Let $Z \in \mathcal{L}_1^C \cup \mathcal{L}_2^C$. As $v \notin Z$, Lemma 37 implies that $B \setminus \{v\}$ is an (X^C, Z)-bs in G_{X^C}.

Property iii. We must prove that $B \setminus \{v\}$ is not an (X^C, Z)-bs in G_{X^C}, for every $Z \in \mathcal{S}^C = \mathcal{S} \cup r_v(\mathcal{L}_2^A)$. Let $Z \in \mathcal{S}^C$. If $Z \in \mathcal{S}$, as $v \notin Z$ (which we know from Property 3) and B is not an (X, Z)-bs in G_X (since $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$), Lemma 37 implies that $B \setminus \{v\}$ is not an (X^C, Z)-bs in G_{X^C}. If $Z \in r_v(\mathcal{L}_2^A)$, then let Z' be such that $Z = Z' \setminus \{v\}$. We know that $B \setminus \{v\}$ is not an (X, Z')-bs in G_X. By Lemma 36, we get that $B \setminus \{v\}$ is not an (X^C, Z)-bs in G_{X^C}.

Property iva. Let us now prove that, for every $v' \in (B \setminus \{v\}) \setminus (B_0 \setminus \{v\})$ there is $Z \in \mathcal{L}_1^C = \mathcal{L}_1(\bar{v})$ such that $(B \setminus \{v\}) \setminus \{v'\}$ is not an (X^C, Z)-bs in G_{X^C}. Since $v \in B_0$, let $v' \in B \setminus B_0$. Since $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, there exists $Z \in \mathcal{L}_1$ such that $B \setminus \{v'\}$ is not an (X, Z)-bs in G_X. As $v \in B \setminus \{v'\}$, this implies that $Z \in \mathcal{L}_1(\bar{v})$. As $v \notin Z$, from Lemma 37 we get that $(B \setminus \{v\}) \setminus \{v'\}$ is not an (X^C, Z)-bs in G_{X^C}.

Property ivb. We now have to prove that for every $v' \in B_0 \setminus \{v\}$, $(B \setminus \{v\}) \setminus \{v'\}$ is not an $(X^C, f^C(v'))$-bs in G_{X^C}. If $B_0 \setminus \{v\} = \emptyset$, we have nothing to prove. Otherwise, let $v' \in B_0 \setminus \{v\}$.

Since $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, there exists $Z \in \mathcal{L}_2$ such that $Z = f^C(v') = f(v')$ and $B \setminus \{v'\}$ is not an (X, Z)-bs in G_X. As $v \in B \setminus \{v'\}$, this implies that $Z \in \mathcal{L}_2(\bar{v})$. As $v \notin Z$, Lemma 37 implies that $(B \setminus \{v\}) \setminus \{v'\}$ is not an (X^C, Z)-bs in G_{X^C}.

Case 2. Suppose that $v \notin B$. Let us prove that $B \vdash (X \setminus \{v\}, B_0, r_v(\mathcal{L}_1), r_v(\mathcal{L}_2), f^C, r_v(\mathcal{S}))$, where $f^C(v') = f(v') \setminus \{v\}$ for every $v' \in B_0$.

Property ii. We first prove that B is an (X^C, Z)-set in G_{X^C}, for every $Z \in \mathcal{L}_1^C \cup \mathcal{L}_2^C = r_v(\mathcal{L}_1) \cup r_v(\mathcal{L}_2)$. Let $Z \in r_v(\mathcal{L}_1) \cup r_v(\mathcal{L}_2)$ where $Z = Z' \setminus \{v\}$ and $Z' \in \mathcal{L}_1 \cup \mathcal{L}_2$. As $v \notin B$ and B is an (X, Z')-bs in G_X, Lemma 36 implies that B is an (X^C, Z)-bs in G_{X^C}.

Property iii. Let us prove that B is not an (X^C, Z)-bs in G_{X^C}, for every $Z \in \mathcal{S}^C = r_v(\mathcal{S})$. Let $Z \in r_v(\mathcal{S})$, where $Z = Z' \setminus \{v\}$ and $Z' \in \mathcal{S}$. As $v \notin B$, and as B is not an (X, Z')-bs in G_X, Lemma 36 implies that B is not an (X^C, Z)-bs in G_{X^C}.

Property iva. We now prove that, for every $v' \in B \setminus B_0$, there exists $Z \in \mathcal{L}_1^C = r_v(\mathcal{L}_1)$ such that $B \setminus \{v'\}$ is not an (X^C, Z)-bs in G_{X^C}. Suppose that $B \setminus B_0 \neq \emptyset$, as otherwise the statement trivially holds. Let $v' \in B \setminus B_0$. Since $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, there exists $Z \in \mathcal{L}_1$ such that $B \setminus \{v'\}$ is not an (X, Z)-bs in G_X. As $v \notin B \setminus \{v'\}$, Lemma 36 implies that $B \setminus \{v'\}$ is not an $(X^C, Z \setminus \{v\})$-bs in G_{X^C}, and $Z \setminus \{v\} \in \mathcal{L}_1^C$.

Property ivb. We must finally prove that, for every $v' \in B_0$, $B \setminus \{v'\}$ is not an $(X^C, f^C(v'))$-bs in G_{X^C}. Let $v' \in B_0$. Since $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, let $Z = f(v')$, where $Z \in \mathcal{L}_2$, such that $B \setminus \{v'\}$ is not an (X, Z)-bs in G_X (by Property ivb). As $v \notin B \setminus \{v'\}$, Lemma 36 implies that $B \setminus \{v'\}$ is not an $(X^C, Z \setminus \{v\})$-bs in G_{X^C}, and $Z \setminus \{v\} = f^C(v')$.

We now focus on the backward implication, and we distinguish again the two cases according to the possible hypothesis. In both cases, remind that our goal is to prove that $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$.

Case 1. Let B with $v \in B$ and suppose that there exist $\mathcal{L}_2^A, \mathcal{L}_2^B$ satisfying the statement of the lemma.

Property ii. Let us prove that B is an (X, Z)-bs in G_X, for every $Z \in \mathcal{L}_1 \cup \mathcal{L}_2$. Let $Z \in \mathcal{L}_1 \cup \mathcal{L}_2$. If $v \in Z$, then as $v \in B$ it follows that B is an (X, Z)-bs in G_X. Otherwise, by Property ii, we get that $B \setminus \{v\}$ is an (X^C, Z)-bs in G_{X^C}. As $v \notin Z$, Lemma 37 implies that B is an (X, Z)-bs in G_X.

Property iii. We now prove that B is not an (X, Z)-bs in G_X, for every $Z \in \mathcal{S}$. Let $Z \in \mathcal{S}$. We know that $B \setminus \{v\}$ is not an (X^C, Z)-bs in G_{X^C}. By hypothesis, $v \notin Z$, and thus Lemma 37 implies that B is not an (X, Z)-bs in G_X.

Property iva. We have to show that, for every $v' \in B \setminus B_0$, there exists $Z \in \mathcal{L}_1$ such that $B \setminus \{v'\}$ is not an (X, Z)-bs in G_X. Assume that $B \setminus B_0 \neq \emptyset$. Let $v' \in B \setminus B_0$. By hypothesis, there exists $Z \in \mathcal{L}_1^C = \mathcal{L}_1(\bar{v})$ such that $(B \setminus \{v\}) \setminus \{v'\}$ is not an (X^C, Z)-bs in G_{X^C}. As $v \notin Z$, Lemma 37 implies that $(B \setminus \{v'\})$ is not an (X, Z)-bs in G_X.

Property ivb. We finally prove that, for every $v' \in B_0$, $B \setminus \{v'\}$ is not an $(X, f(v'))$-bs in G_X. Let $v' \in B_0$ and let $Z = f(v')$. Suppose first that $v' = v$. By Property 2 we know that $Z \in \mathcal{L}_2^A$, and by Property 1 we know that $v \in Z$, implying then that there exists Z' such that $Z = Z' \cup \{v\}$. As $r_v(\mathcal{L}_2^A) \subseteq \mathcal{S}^C$, it follows that $B \setminus \{v\}$ is not an (X^C, Z')-bs in G_{X^C}. By Lemma 36, $B \setminus \{v\}$ is not an (X, Z)-bs in G_X. Suppose now that $v' \neq v$. By hypothesis, $(B \setminus \{v\}) \setminus \{v'\}$ is not an (X^C, Z)-bs in G_{X^C}, and $Z \in \mathcal{L}_2(\bar{v})$. As $v \notin Z$, by Lemma 37 $B \setminus \{v'\}$ is not an (X, Z)-bs in G_X.

Case 2. Let B with $v \notin B$ and $B \vdash (X \setminus \{v\}, B_0, r_v(\mathcal{L}_1), r_v(\mathcal{L}_2), f^C, r_v(\mathcal{S}))$, where $f^C(v') = f(v') \setminus \{v\}$ for every $v' \in B_0$.

Property ii. Let us prove that B is an (X, Z)-bs in G_X, for every $Z \in \mathcal{L}_1 \cup \mathcal{L}_2$. Let $Z \in \mathcal{L}_1 \cup \mathcal{L}_2$. As $v \notin B$ and as B is an $(X^C, Z \setminus \{v\})$-bs in G_{X^C}, Lemma 36 implies that B

is an (X,Z)-bs in G_X.

Property iii. We now prove that B is not an (X,Z)-bs in G_X, for every $Z \in \mathcal{S}$. Let $Z \in \mathcal{S}$. As $v \notin B$ and as B is not an $(X^C, Z \setminus \{v\})$-bs in G_{X^C}, Lemma 36 implies that B is not an (X,Z)-bs in G_X.

Property iva. We have to show that, for every $v' \in B \setminus B_0$, there exists $Z \in \mathcal{L}_1$ such that $B \setminus \{v'\}$ is not an (X,Z)-bs in G_X. If $B \setminus B_0 = \emptyset$, then we have nothing to prove. Let $v' \in B \setminus B_0$. By hypothesis, there exists $Z \in \mathcal{L}_1^C$ such that $B \setminus \{v'\}$ is not an (X^C, Z)-bs in G_{X^C}. Let $Z' \in \mathcal{L}_1$ such that $Z = Z' \setminus \{v\}$. As $v \notin B \setminus \{v'\}$, Lemma 36 implies that $B \setminus \{v'\}$ is not an (X, Z')-bs in G_X.

Property ivb. We finally prove that, for every $v' \in B_0$, $B \setminus \{v'\}$ is not an $(X, f(x))$-bs in G_X. Let $v' \in B_0$ and let $Z = f(v')$, where $Z \in \mathcal{L}_2$. Recall that $f^C(v') = Z \setminus \{v\}$. By hypothesis, $B \setminus \{v'\}$ is not an $(X^C, Z \setminus \{v\})$-bs in G_{X^C}. As $v \notin B \setminus \{v'\}$, Lemma 36 implies that $B \setminus \{v'\}$ is not an (X,Z)-bs in G_X. ◻

4.4 Forget node

Let us start with some preliminaries related to the notion of *criticality*.

▶ **Definition 39.** *Let G be a graph, $X \subseteq V(G)$, $Z \subseteq X$ such that Z is an* is*, and $v \in V(G)$. We say that (X,Z) is*

- *v-critical in G if for every maximum (X,Z)-is I in G, $v \in I$,*
- *$\bar{v}$-critical in G if for every maximum (X,Z)-is I in G, $v \notin I$, and*
- *v-mixed in G if there exists a maximum (X,Z)-is I in G with $v \in I$ and there exists a maximum (X,Z)-is I' in G with $v \notin I$.*

Given $v \in V(G)$ and a set $\mathcal{R} \subseteq 2^X$ such that for each $Z \in \mathcal{R}$, Z is an is*, we denote*

- *$\mathcal{R}^{(v,X)} = \{Z \in \mathcal{R} \mid (X,Z)$ is v-critical in $G_X\}$,*
- *$\mathcal{R}^{(\bar{v},X)} = \{Z \in \mathcal{R} \mid (X,Z)$ is $\bar{v}$-critical in $G_X\}$,*
- *$\mathcal{R}^{(*v,X)} = \{Z \in \mathcal{R} \mid (X,Z)$ is v-mixed in $G_X\}$, and*
- *$a_v(\mathcal{R}) = \{Z \cup \{v\} \mid Z \in \mathcal{R}\}$.*

▶ **Lemma 40.** *Let G be a graph, $X \subseteq V(G)$, and $Z \subseteq X$ such that Z is an* is *and $v \in V(G)$. Deciding whether (X,Z) is v-critical in G, $\bar{v}$-critical in G or v-mixed in G can be done in time $\mathcal{O}^*(2^{\mathsf{tw}(G)})$.*

Proof: If $v \in Z$, then (X,Z) is by definition v-critical in G, and if $v \in X \setminus Z$ then (X,Z) is by definition $\bar{v}$-critical G. Suppose now that $v \notin X$. Then, observe that

- (X,Z) is v-critical in G if and only if $\alpha_{(X \cup \{v\}, Z)}(G) < \alpha_{(X,Z)}(G)$,
- (X,Z) is $\bar{v}$-critical in G if and only if $\alpha_{(X \cup \{v\}, Z \cup \{v\})}(G) < \alpha_{(X,Z)}(G)$, and
- (X,Z) is v-mixed in G if and only if $\alpha_{(X \cup \{v\}, Z \cup \{v\})}(G) = \alpha_{(X \cup \{v\}, Z)}(G) = \alpha_{(X,Z)}(G)$.

Let us now prove that, for every (X',Z') such that $Z' \subseteq X'$ and Z' is an is, $\alpha_{(X',Z')}(G)$ can be computed in time $\mathcal{O}^*(2^{\mathsf{tw}(G)})$. Indeed, observe that $\alpha_{(X',Z')}(G) = |Z'| + \alpha(G \setminus (X' \cup N(Z')))$. As $\mathsf{tw}(G \setminus (X' \cup N(Z'))) \leq \mathsf{tw}(G')$ and $\alpha(G)$ can be computed in time $\mathcal{O}^*(2^{\mathsf{tw}(G)})$ [14], we get the desired result. ◻

Before proving Lemma 45 corresponding to the forget case, let us first prove the following lemmas, where we assume that $X \in \mathcal{B}$ is a forget node and X^C is the child of X with $X^C = X \cup \{v\}$, for some vertex $v \in X_C$.

▶ **Lemma 41.** *Let $I \subseteq V(G)$ such that $v \in I$ and let $Z \subseteq X$ such that Z is an* is*. The following claims hold:*

- *If I is a maximum (X,Z)-*is *in G_X, then I is a maximum $(X^C, Z \cup \{v\})$-*is *in G_{X^C}.*
- *If (X,Z) is not $\bar{v}$-critical in G_X, then I is a maximum (X,Z)-*is *in G_X if and only if I is a maximum $(X^C, Z \cup \{v\})$-*is *in G_{X^C}.*

Proof: For the first item, let I be a maximum (X,Z)-is in G_X. As $v \in I$, $I \cap X^C = Z \cup \{v\}$, and I is also an $(X^C, Z \cup \{v\})$-is in G_{X^C}. Let I' be a maximum $(X^C, Z \cup \{v\})$-is in G_{X^C}. As I' is also an (X,Z)-is in G_X, $|I| \geq |I'|$, and thus I is a maximum $(X^C, Z \cup \{v\})$-is in G_{X^C}.

For the second item, the sufficiency is already proved in the first item. For the backward implication, let I be a maximum $(X^C, Z \cup \{v\})$-is in G_{X^C}. Observe first that I is an (X,Z)-is in G_X. Let I' be a maximum (X,Z)-is in G_X such that $v \in I'$, which exists as (X,Z) is not $\bar{v}$-critical in G_X. By the previous property, I' is a (maximum) $(X^C, Z \cup \{v\})$-is in G_{X^C}, implying $|I| \geq |I'|$ and the desired result. ☐

▶ **Lemma 42.** *Let $I \subseteq V(G)$ such that $v \notin I$ and let $Z \subseteq X$ such that Z is an* is*. The following claims hold:*

- *If I is a maximum (X,Z)-*is *in G_X, then I is a maximum (X^C, Z)-*is *in G_{X^C}.*
- *If (X,Z) is not v-critical in G_X, then I is a maximum (X,Z)-*is *in G_X if and only if I is a maximum (X^C, Z)-*is *in G_{X^C}.*

Proof: For the first item, let I be a maximum (X,Z)-is in G_X. As $v \notin I$, $I \cap X^C = Z$, hence I is also an (X^C, Z)-is in G_{X^C}. Let I' be a maximum (X^C, Z)-is in G_{X^C}. As I' is also an (X,Z)-is in G_X, $|I| \geq |I'|$, and thus I is a maximum (X^C, Z)-is in G_{X^C}.

For the second item, again we only need to prove the backward implication. Let I be a maximum (X^C, Z)-is in G_{X^C}. Observe first that I is an (X,Z)-is in G_X. Let I' be a maximum (X,Z)-is in G_X such that $v \notin I'$, which exists as (X,Z) is not v-critical in G_X. By the first item, I' is a (maximum) (X^C, Z)-is in G_{X^C}, implying $|I| \geq |I'|$ and the desired result. ☐

▶ **Lemma 43.** *Let $Z \subseteq X$ where Z is an* is*. The following claims hold:*

- *If (X,Z) is v-critical in G_X, then for each $B \subseteq V(G_X)$, B is an (X,Z)-*bs *in G_X if and only if B is an $(X^C, Z \cup \{v\})$-*bs *in G_{X^C}.*
- *If (X,Z) is $\bar{v}$-critical in G_X, then for each $B \subseteq V(G_X)$, B is an (X,Z)-*bs *in G_X if and only if B is an (X^C, Z)-*bs *in G_{X^C}.*
- *If (X,Z) is v-mixed in G_X, then for each $B \subseteq V(G_X)$, B is an (X,Z)-*bs *in G_X if and only if B is an $(X^C, Z \cup \{v\})$-*bs *in G_{X^C} and B is an (X^C, Z)-*bs *in G_{X^C}.*

Proof: For the first item, let $Z \subseteq X$ such that (X,Z) is v-critical in G_X.

For the forward implication, suppose that B is an (X,Z)-bs in G_X. Let I be a maximum $(X^C, Z \cup \{v\})$-is in G_{X^C}. As $v \in I$ and as (X,Z) is not $\bar{v}$-critical in G_X, by Lemma 41, I is also a maximum (X,Z)-is in G_X, implying that $I \cap B \neq \emptyset$.

For the backward implication, let B be an $(X^C, Z \cup \{v\})$-bs G_{X^C}. Let I be a maximum (X,Z)-is in G_X. As (X,Z) is v-critical in G_X, we know that $v \in I$. By Lemma 41, I is also a maximum $(X^C, Z \cup \{v\})$-is in G_{X^C}, implying that $I \cap B \neq \emptyset$.

For the second item, let $Z \subseteq X$ such that (X,Z) is $\bar{v}$-critical in G_X.

For the forward implication, suppose that B is an (X,Z)-bs in G_X. Let I be a maximum (X^C, Z)-is in G_{X^C}. As $v \notin I$ and as (X,Z) is not v-critical in G_X, by Lemma 42, I is also a maximum (X,Z)-is in G_X, implying that $I \cap B \neq \emptyset$.

For the backward implication, let B be an (X^C, Z)-bs G_{X^C}. Let I be a maximum (X,Z)-is in G_X. As (X,Z) is $\bar{v}$-critical in G_X, we know that $v \notin I$. By Lemma 42, I is also a maximum (X^C, Z)-is in G_{X^C}, implying that $I \cap B \neq \emptyset$.

For the third item, let $Z \subseteq X$ such that (X, Z) is v-mixed in G_X.

For the forward implication, assume that B is an (X, Z)-bs in G_X. Let I_1 be a maximum $(X^C, Z \cup \{v\})$-is in G_{X^C} and I_2 be a maximum (X^C, Z)-is in G_{X^C}. As (X, Z) is both not v-critical and not $\bar{v}$-critical in G_X, by Lemmas 41 and 42 we now that both I_1 and I_2 are maximum (X, Z)-is in G_X, implying $B \cap I_1 \neq \emptyset$ and $B \cap I_2 \neq \emptyset$.

For the backward implication, let finally B be an $(X^C, Z \cup \{v\})$-bs in G_{X^C} and an (X^C, Z)-bs in G_{X^C}. Let I be a maximum (X^C, Z)-is in G_{X^C}. If $v \in I$, by Lemma 41, I is also a maximum $(X^C, Z \cup \{v\})$-is in G_{X^C}, implying $I \cap B \neq \emptyset$, and if $v \notin I$, by Lemma 42, I is also a maximum (X^C, Z)-is in G_{X^C}, implying $I \cap B \neq \emptyset$ as well. ◻

▶ **Lemma 44.** *Let $\mathcal{L} \subseteq 2^X$ where for each $Z \in \mathcal{L}$, Z is an* is*. For every $B \subseteq G_X$, B is an (X, Z)-*bs *in G_X for every $Z \in \mathcal{L}$ if and only if B is an (X^C, Z)-*bs *in G_{X^C} for every $Z \in a_v(\mathcal{L}^{(v,X)}) \cup \mathcal{L}^{(\bar{v},X)} \cup a_v(\mathcal{L}^{(*v,X)}) \cup \mathcal{L}^{(*v,X)}$.*

Proof: For the forward implication, suppose that B is an (X, Z)-bs in G_X, for every $Z \in \mathcal{L}$. Let $Z \in a_v(\mathcal{L}^{(v,X)})$ (resp. $Z \in a_v(\mathcal{L}^{(*v,X)})$), implying that $Z = Z' \cup \{v\}$ with $Z' \in \mathcal{L}^{(v,X)}$ (resp. $Z' \in \mathcal{L}^{(*v,X)}$). Observe that for every $Z \in \mathcal{L}$, $v \notin Z$, implying that $v \notin Z'$ and thus that we also have $Z' = Z \setminus \{v\}$. This implies that (X, Z') is v-critical (resp. v-mixed) in G_X. By hypothesis, B is an (X, Z')-bs in G_X, implying, as (X, Z') is v-critical (resp. v-mixed) in G_X, that B is an (X^C, Z)-bs in G_{X^C} by Lemma 43. Let now $Z \in \mathcal{L}^{(\bar{v},X)}$ (resp. $Z \in \mathcal{L}^{(*v,X)}$), implying that (X, Z) is $\bar{v}$-critical (resp. v-mixed) in G_X. By hypothesis, B is an (X, Z)-bs in G_X, implying, as (X, Z) is $\bar{v}$-critical (resp. v-mixed) in G_X, that B is an (X^C, Z)-bs in G_{X^C} by Lemma 43.

For the backward implication, suppose that B is an (X^C, Z)-bs in G_{X^C} for every $Z \in a_v(\mathcal{L}^{(v,X)}) \cup \mathcal{L}^{(\bar{v},X)} \cup a_v(\mathcal{L}^{(*v,X)}) \cup \mathcal{L}^{(*v,X)}$. Let $Z \in \mathcal{L}$. If $Z \in \mathcal{L}^{(v,X)}$, then there exists $Z' \in a_v(\mathcal{L}^{(v,X)})$ such that $Z' = Z \cup \{v\}$. By hypothesis, B is an (X^C, Z')-bs in G_{X^C}, implying, as (X, Z) is v-critical in G_X, that B is an (X, Z)-bs in G_X by Lemma 43. If $Z \in \mathcal{L}^{(\bar{v},X)}$, then by hypothesis, B is an (X^C, Z)-bs in G_{X^C}, implying, as (X, Z) is $\bar{v}$-critical in G_X, that B is an (X, Z)-bs in G_X by Lemma 43. If $Z \in \mathcal{L}^{(*v,X)}$, then by hypothesis B is an $(X^C, Z \cup \{v\})$-bs in G_{X^C} and B is an (X^C, Z)-bs in G_{X^C}, implying, as (X, Z) is v-mixed in G_X, that B is an (X, Z)-bs in G_X by Lemma 43. ◻

We are now ready to state the main lemma of this section.

▶ **Lemma 45.** *Let $(X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S}) \in \mathcal{E}$ where $X \in \mathcal{B}$ is a forget node and X^C is the child of X with $X^C = X \cup \{v\}$. For each $B \subseteq V(G_X)$, $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$ if and only if one of the following two cases holds:*

***Case 1**: there exists $Z^* \in \mathcal{L}_1$ such that*

1. *(X, Z^*) is not $\bar{v}$-critical in G_X,*
2. *for each $Z \in \mathcal{S}$, (X, Z) is not v-critical in G_X,*
3. *for each $v' \in B_0$, $(X, f(v'))$ is not v-critical in G_X, and*
4. *$B \vdash (X \cup \{v\}, B_0 \cup \{v\}, \mathcal{L}_1^C, \mathcal{L}_2^C, f^C, \mathcal{S}^C)$, where*

- *$\mathcal{L}_1^C = a_v(\mathcal{L}_1^{(v,X)}) \cup \mathcal{L}_1^{(\bar{v},X)} \cup a_v(\mathcal{L}_1^{(*v,X)}) \cup \mathcal{L}_1^{(*v,X)}$,*
- *$\mathcal{L}_2^C = a_v(\mathcal{L}_2^{(v,X)}) \cup \mathcal{L}_2^{(\bar{v},X)} \cup a_v(\mathcal{L}_2^{(*v,X)}) \cup \mathcal{L}_2^{(*v,X)} \cup \{Z^* \cup \{v\}\}$,*
- *$f^C : B_0 \cup \{v\} \to \mathcal{L}_2^C$ is such that*

$$f^C(v') = \begin{cases} Z^* \cup \{v\} & \text{, if } v' = v, \\ f(v') & \text{, otherwise, and} \end{cases}$$

- $\mathcal{S}^C = \mathcal{S}$.

Case 2: *there exist $\mathcal{S}^A, \mathcal{S}^B$, and f^C such that*

1. $\mathcal{S}^{(*v,X)} = \mathcal{S}^A \uplus \mathcal{S}^B$ *and*

2. $B \vdash (X \cup \{v\}, B_0, \mathcal{L}_1^C, \mathcal{L}_2^C, f^C, \mathcal{S}^C)$*, where*

- $\mathcal{L}_1^C = a_v(\mathcal{L}_1^{(v,X)}) \cup \mathcal{L}_1^{(\bar{v},X)} \cup a_v(\mathcal{L}_1^{(*v,X)}) \cup \mathcal{L}_1^{(*v,X)}$,
- $\mathcal{L}_2^C = a_v(\mathcal{L}_2^{(v,X)}) \cup \mathcal{L}_2^{(\bar{v},X)} \cup a_v(\mathcal{L}_2^{(*v,X)}) \cup \mathcal{L}_2^{(*v,X)}$,
- $f^C : B_0 \to \mathcal{L}_2^C$ *is such that*

$$f^C(v') = \begin{cases} f(v') \cup \{v\} & \text{, if } f(v') \in \mathcal{L}_2^{(v,X)}, \\ f(v') & \text{, if } f(v') \in \mathcal{L}_2^{(\bar{v},X)}, \end{cases}$$

  otherwise $f^C(v') \in \{f(v'), f(v') \cup \{v\}\}$*, and*
- $\mathcal{S}^C = a_v(\mathcal{S}^{(v,X)}) \cup \mathcal{S}^{(\bar{v},X)} \cup a_v(\mathcal{S}^A) \cup \mathcal{S}^B$.

Proof: Observe first that in both cases, for every $Z \in \mathcal{L}_1^C \cup \mathcal{L}_2^C \cup \mathcal{S}^C$, Z is an is as required in the definition of $\mathcal{E}$. Indeed, for each $Z \in \mathcal{L}_1 \cup L_2 \cup \mathcal{S}$, we only add $Z \cup \{v\}$ to $\mathcal{L}_1^C \cup \mathcal{L}_2^C \cup \mathcal{S}^C$ when Z is not $\bar{v}$-critical in G_X, implying that $Z \cup \{v\}$ is an is.

For the forward implication, suppose that $B \subseteq V(G_X)$ is such that $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, and let us distinguish two cases.

Suppose first that $v \in B$. In this case, we will prove that all statements corresponding to Case 1 hold. Recall that $X = X^C \setminus \{v\}$ and $B_0 = B \cap X$. Thus $v \in B \setminus B_0$. Since $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, Property iva in Definition 26 implies that there exists $Z^* \in \mathcal{L}_1$ such that $B \setminus \{v\}$ is not an (X, Z^*)-bs in G_X, implying that there exists a maximum (X, Z^*)-is I^* in G_X such that $I^* \cap (B \setminus \{v\}) = \emptyset$. Moreover, $v \in I^*$ as otherwise, $I^* \cap B = \emptyset$, contradicting the fact that B is an (X, Z^*)-bs in G_X. This implies Property 1 of Case 1, i.e. (X, Z^*) is not $\bar{v}$-critical in G_X.

Let $v' \in B_0$, $Z = f(v')$, and $B' = B \setminus \{v'\}$. As $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, Property ivb implies that B' is not an (X, Z)-bs in G_x. Thus, there exists a maximum (X, Z)-is I in G_X such that $I \cap B' = \emptyset$. As $v \in B'$, we deduce that (X, Z) is not v-critical in G_X, implying Property 3 of Case 1. Let us now prove Property 4 and along the proof we will verify that Property 2 is also satisfied.

Thus, let us now check all properties of Definition 26 to prove that $B \vdash (X \cup \{v\}, B_0 \cup \{v\}, \mathcal{L}_1^C, \mathcal{L}_2^C, f^C, \mathcal{S}^C)$, where $\mathcal{L}_1^C$, $\mathcal{L}_2^C$, f^C, and $\mathcal{S}^C$ are defined as in Case 1.

Property ii. Let us prove that B is an (X^C, Z)-bs in G_{X^C}, for each $Z \in \mathcal{L}_1^C \cup \mathcal{L}_2^C$. By Lemma 44, for each $Z \in \mathcal{L}_1^C \cup (\mathcal{L}_2^C \setminus \{Z^* \cup \{v\}\})$, we know that B is an (X^C, Z)-bs in G_{X^C}. Moreover, recall that $Z^* \in \mathcal{L}_1$ and that $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$. Thus, B is an (X, Z^*)-bs in G_X and, as (X, Z^*) is either v-critical or v-mixed in G_X, this implies by Lemma 43 that B is an $(X^C, Z^* \cup \{v\})$-bs in G_{X^C}.

Property iii. We now prove that B is not an (X^C, Z)-bs in G_{X^C}, for each $Z \in \mathcal{S}^C = \mathcal{S}$. Let $Z \in \mathcal{S}$. As $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, B is not an (X, Z)-bs in G_X. Since $v \in B$, it follows that (X, Z) is not v-critical in G_X, implying Property 2 of Case 1. If (X, Z) is $\bar{v}$-critical in G_X, then by Lemma 43, we get that B is not an (X^C, Z)-bs in G_{X^C}. If (X, Z) is v-mixed in G_X, since B is not an (X, Z)-bs in G_X, by Lemma 43 we get that B is not an (X^C, Z)-bs in G_{X^C} or B is not an $(X^C, Z \cup \{v\})$-bs in G_{X^C}. The latter case is not possible as $v \in B$, and thus we get the desired property.

Property iva. Let us prove that, for each $v' \in B \setminus (B_0 \cup \{v\})$, there is $Z \in \mathcal{L}_1^C$ such that $B \setminus \{v'\}$ is not an (X^C, Z)-bs in G_{X^C}. Let $v' \in B \setminus (B_0 \cup \{v\})$. As $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, there exists $Z \in \mathcal{L}_1$ such that $B' = B \setminus \{v'\}$ is not an (X, Z)-bs in G_X. Notice that as $v \in B'$, (X, Z) is not v-critical in G_X. If (X, Z) is $\bar{v}$-critical in G_X, then by Lemma 43, B' is not an (X^C, Z)-bs in G_{X^C}, and we are done as $Z \in \mathcal{L}_1^{(\bar{v},X)} \subseteq \mathcal{L}_1^C$. If (X, Z) is v-mixed in G_X, then by Lemma 43, B' is not an (X^C, Z)-bs in G_{X^C} or B' is not an $(X^C, Z \cup \{v\})$-bs in G_{X^C}. Again, this latter case is not possible as $v \in B'$. Thus, we conclude the proof as $Z \in \mathcal{L}_1^{(*v,X)} \subseteq \mathcal{L}_1^C$.

Property ivb. To finish this case, let us prove that for each $v' \in B_0 \cup \{v\}$, $B \setminus \{v'\}$ is not an $(X^C, f^C(v'))$-bs in G_{X^C}. Let $v' \in B_0^C = B_0 \cup \{v\}$. Let $B' = B \setminus \{v'\}$. Let us first consider the case $v' = v$. In this case, remind that $f^C(v') = Z^* \cup \{v\}$, where, as chosen above, $Z^* \in \mathcal{L}_1$ such that $B \setminus \{v\}$ is not an (X, Z^*)-bs in G_X. Then let us consider I^* defined above, i.e. a maximum (X, Z^*)-is in G_X such that $I^* \cap (B \setminus \{v\}) = \emptyset$. As $v \in I^*$, according to Lemma 41, I^* is a maximum $(X^C, Z^* \cup \{v\})$-is in G_{X^C}, and $B' \cap I^* = \emptyset$. Suppose now that $v' \neq v$ and remind that, in this case, $f^C(v') = f(v')$. Then, since $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$ and $v' \in B_0$, we have that B' is not an (X, Z)-bs in G_X, where $Z = f(v') \in \mathcal{L}_2$. As $v \in B'$, we deduce that (X, Z) is not v-critical in G_x. If (X, Z) is v-mixed in G_X, then by Lemma 43, B' is not an (X^C, Z)-bs in G_{X^C} or B' is not an $(X^C, Z \cup \{v\})$-bs in G_{X^C}. This last case is again not possible as $v \in B'$. Thus, we deduce that B' is not an (X^C, Z)-bs. If (X, Z) is $\bar{v}$-critical in G_X, then by Lemma 43, we also get that B' is not an (X^C, Z)-bs in G_{X^C}.

Suppose now $v \notin B$. We now prove that Case 2 of lemma's statement holds. Since $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, we have that, for each $Z \in \mathcal{S}^{(*v,X)}$, B is not an (X, Z)-bs in G_X. By Lemma 43 we get that either B is not an (X^C, Z)-bs in G_{X^C}, in which case we add Z to $\mathcal{S}^B$, and if it is not the case then B is not an $(X^C, Z \cup \{v\})$-bs in G_{X^C}, in which case we add Z to $\mathcal{S}^A$. It remains to define the function f^C for $v' \neq v$ such that $f(v') \in \mathcal{L}_2^{(*v,X)}$. Let $v' \neq v$ such that $f(v') \in \mathcal{L}_2^{(*v,X)}$, and let $Z = f(v')$. Since $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, $B \setminus \{v'\}$ is not an (X, Z)-bs in G_X. By Lemma 43, as (X, Z) is mixed in G_X, we get that either $B \setminus \{v'\}$ is not an (X^C, Z)-bs in G_{X^C}, in which case we define $f^C(v') = Z$, and if it is not the case then $B \setminus \{v'\}$ is not an $(X^C, Z \cup \{v\})$-bs in G_{X^C}, in which case we define $f^C(v') = Z \cup \{v\}$. Let us now prove that $B \vdash (X^C, B_0, \mathcal{L}_1^C, \mathcal{L}_2^C, f^C, \mathcal{S}^C)$ where $\mathcal{L}_1^C$, $\mathcal{L}_2^C$, and $\mathcal{S}^C$ are defined as in the Case 2 of the lemma's statement, by verifying that the required properties in Definition 26 are satisfied. To prove Property ii, one should argue that B is an (X^C, Z)-bs in G_{X^C}, for each $Z \in \mathcal{L}_1^C \cup \mathcal{L}_2^C$ where $\mathcal{L}_1^C = a_v(\mathcal{L}_1^{(v,X)}) \cup \mathcal{L}_1^{(\bar{v},X)} \cup a_v(\mathcal{L}_1^{(*v,X)}) \cup \mathcal{L}_1^{(*v,X)}$ and $\mathcal{L}_2^C = a_v(\mathcal{L}_2^{(v,X)}) \cup \mathcal{L}_2^{(\bar{v},X)} \cup a_v(\mathcal{L}_2^{(*v,X)}) \cup \mathcal{L}_2^{(*v,X)}$. It is immediate using Lemma 44 and the hypothesis that $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$.

Property iii. Let us now prove that B is not an (X^C, Z)-bs in G_{X^C}, for each $Z \in \mathcal{S}^C = a_v(\mathcal{S}^{(v,X)}) \cup \mathcal{S}^{(\bar{v},X)} \cup a_v(S^A) \cup a_v(S^B)$. Let $Z \in \mathcal{S}^C$. If $Z \in a_v(\mathcal{S}^{(v,X)})$ then $Z = Z' \cup \{v\}$ where $Z' \in \mathcal{S}^{(v,X)}$. Since $B \vdash (X^C, B_0, \mathcal{L}_1^C, \mathcal{L}_2^C, f^C, \mathcal{S}^C)$, we know that B is not an (X, Z')-bs in G_X. As (X, Z') is v-critical in G_X, by Lemma 43 we know that B is not an $(X^C, Z' \cup \{v\})$-bs in G_{X^C}. If $Z \in \mathcal{S}^{(\bar{v},X)}$ then by the hypothesis $B \vdash (X^C, B_0, \mathcal{L}_1^C, \mathcal{L}_2^C, f^C, \mathcal{S}^C)$, we know that B is not an (X, Z)-bs in G_X. As (X, Z) is $\bar{v}$-critical in G_X, by Lemma 43 we know that B is not an (X^C, Z)-bs in G_{X^C}. If $Z \in a_v(\mathcal{S}^A) \cup \mathcal{S}^B$, then by definition of $\mathcal{S}^A$ and $\mathcal{S}^B$, B is not an (X^C, Z)-bs in G_{X^C}.

Property iva. We must now prove that, for each $v' \in B \setminus B_0$, there is $Z \in \mathcal{L}_1^C = a_v(\mathcal{L}_1^{(v,X)}) \cup \mathcal{L}_1^{(\bar{v},X)} \cup a_v(\mathcal{L}_1^{(*v,X)}) \cup \mathcal{L}_1^{(*v,X)}$ such that $B \setminus \{v'\}$ is not an (X^C, Z)-bs in G_{X^C}. Recall that $v \notin B$ and let $v' \in B \setminus (B_0 \cup \{v\})$. As $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, there exists $Z \in \mathcal{L}_1$ such that $B' = B \setminus \{v'\}$ is not an (X, Z)-bs in G_X. By Lemma 44 with the list $\mathcal{L} = \{Z\}$, there exists $Z' \in \mathcal{L}_1^C$ such that B' is not an (X^C, Z')-bs in G_{X^C}.

Property ivb. Let us finally prove that, for each $v' \in B_0^C = B_0$, $B \setminus \{v'\}$ is not an $(X^C, f^C(v'))$-bs in G_{X^C}. Let $v' \in B_0^C$. As $v' \in B_0$, by the hypothesis $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$ we know that $B' = B \setminus \{v'\}$ is not an (X, Z)-bs in G_X, where $Z = f(v')$ with $Z \in \mathcal{L}_2$. If (X, Z) is $\bar{v}$-critical in G_X, then by Lemma 43, we also get that B' is not an (X^C, Z)-bs in G_{X^C}, and we are done as $Z = f^C(v')$. If (X, Z) is v-critical in G_X, then by Lemma 43, we also get that B' is not an $(X^C, Z \cup \{v\})$-bs in G_{X^C}, and we are done as $Z \cup \{v\} = f^C(v')$. Finally, (X, Z) is v-mixed then $Z \in \mathcal{L}_2^{(*v,X)}$ and by the definition of $f^C(v')$ we get that B' is not an $(X^C, f(v'))$-bs in G_{X^C}.

For the backward implication, let us prove that $B \vdash (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S})$, by distinguishing again both cases in the statement of the lemma.

Case 1. Suppose that there exist $Z^* \in \mathcal{L}_1$ as required in Case 1. Property ii follows directly from Lemma 44. Let us verify that the other properties of Definition 26 are also verified.

Property iii. Let us prove that B is not (X, Z)-bs in G_X, for each $Z \in \mathcal{S} = \mathcal{S}^C$. Let $Z \in \mathcal{S}$. By hypothesis, $B \vdash (X^C, B_0^C, \mathcal{L}_1^C, \mathcal{L}_2^C, f^C, \mathcal{S}^C)$ and thus B is not an (X^C, Z)-bs in G_{X^C}. Consequently, there exists a maximum (X^C, Z)-is I in G_{X^C} such that $I \cap B = \emptyset$. In addition, since $B_0^C = B_0 \cup \{v\} \subseteq B$, we have that $v \notin I$. As (X, Z) is not v-critical in G_X, we get by Lemma 42 that I is a maximum (X, Z)-is in G_X.

Property iva. We now argue that, for each $v' \in B \setminus B_0$, there is $Z \in \mathcal{L}_1$ such that $B \setminus \{v'\}$ is not an (X, Z)-bs in G_X. Let $v' \in B \setminus B_0$. If $v' = v$, then as $v \in B_0^C$, by definition of f^C we get that $B \setminus \{v\}$ is not an $(X^C, Z^* \cup \{v\})$-bs in G_{X^C}. As (X, Z^*) is either v-critical or v-mixed in G_X, in both cases by Lemma 43 we get that $B \setminus \{v\}$ is not an (X, Z^*)-bs in G_X. As by hypothesis $Z^* \in \mathcal{L}_1$, this implies Property iva. Suppose now $v' \neq v$. Since $B \vdash (X^C, B_0^C, \mathcal{L}_1^C, \mathcal{L}_2^C, f^C, \mathcal{S}^C)$, we know that there exists $Z \in \mathcal{L}_1^C$ such that $B \setminus \{v'\}$ is not an (X^C, Z)-bs in G_{X^C}. By Lemma 44, there exists $Z \in \mathcal{L}_1$ such that $B \setminus \{v'\}$ is not an (X, Z)-bs in G_X.

Property ivb. Finally, we prove that for each $v' \in B_0$, $B \setminus \{v'\}$ is not an $(X, f(v'))$-bs in G_X. Let $v' \in B_0$ and let $Z = f(v')$. Recall that $B_0 = B \cap X$ and thus $v \notin B_0$, implying that $v \neq v'$. By definition, we thus have $f^C(v') = f(v')$. Since $B \vdash (X^C, B_0^C, \mathcal{L}_1^C, \mathcal{L}_2^C, f^C, \mathcal{S}^C)$, we know that $B' = B \setminus \{v'\}$ is not an (X^C, Z)-bs in G_{X^C}. By Property 3, (X, Z) is not v-critical in G_X. As (X, Z) is $\bar{v}$-critical or v-mixed in G_X, then by Lemma 43, B' is not an (X, Z)-bs in G_X.

Case 2. Suppose that there exist $\mathcal{S}^A, \mathcal{S}^B$, and f^C as required in Case 2. Property ii follows again directly from Lemma 44.

Property iii. Let us prove that B is not (X, Z)-bs in G_X, for each $Z \in \mathcal{S}$. Let $Z \in \mathcal{S}$. Recall that $\mathcal{S}^C = a_v(\mathcal{S}^{(v,X)}) \cup \mathcal{S}^{(\bar{v},X)} \cup a_v(\mathcal{S}^A) \cup \mathcal{S}^B$ and that $B \vdash (X^C, B_0, \mathcal{L}_1^C, \mathcal{L}_2^C, f^C, \mathcal{S}^C)$ as claimed in Case 2. If $Z \in \mathcal{S}^{(v,X)}$, then B is not an $(X^C, Z \cup \{v\})$-bs in G_{X^C}, and by Lemma 43, as (X, Z) is v-critical in G_X, B is not an (X, Z)-bs in G_X. If $Z \in \mathcal{S}^{(\bar{v},X)}$, then B is not an (X^C, Z)-bs in G_{X^C}, and by Lemma 43, as (X, Z) is $\bar{v}$-critical in G_X, B is not an (X, Z)-bs in G_X. It remains to treat the case where $Z \in \mathcal{S}^{(*v,X)} = \mathcal{S}^A \cup \mathcal{S}^B$. In this case, if $Z \in \mathcal{S}^A$ then B is not an $(X^C, Z \cup \{v\})$-bs in G_{X^C}, or ($Z \in \mathcal{S}^B$) B is not an (X^C, Z)-bs in G_{X^C}. In both cases, as (X, Z) is v-mixed in G_X, by Lemma 43 B is not an (X, Z)-bs in G_X.

Property iva. We now show that, for each $v' \in B \setminus B_0$, there is $Z \in \mathcal{L}_1$ such that $B \setminus \{v'\}$ is not an (X, Z)-bs in G_X. Let $v' \in B \setminus B_0$. Recall that, since $B \vdash (X^C, B_0, \mathcal{L}_1^C, \mathcal{L}_2^C, f^C, \mathcal{S}^C)$, $B \cap X^C = B_0 \subseteq X$ and $\{v\} = X^C \setminus X$. As $v \notin B$, we deduce $v' \neq v$. By hypothesis, we know that there exists $Z \in \mathcal{L}_1^C$ such that $B \setminus \{v'\}$ is not an (X^C, Z)-bs in G_{X^C}. By Lemma 44, there exists $Z \in \mathcal{L}_1$ such that $B \setminus \{v'\}$ is not an (X, Z)-bs in G_X.

Property ivb. Finally, we prove that for each $v' \in B_0$, $B \setminus \{v'\}$ is not an $(X, f(v'))$-bs in G_X. Let $v' \in B_0$, $Z = f(v')$ and $Z' = f^C(v')$. By hypothesis, we know that $B' = B \setminus \{v'\}$ is not an (X^C, Z')-bs in G_{X^C}. If $Z \in \mathcal{L}_2^{(v,X)}$, then by definition of f^C we have $Z' = Z \cup \{v\}$, and by Lemma 43 we get that B' is not an (X, Z)-bs in G_X. If $Z \in \mathcal{L}_2^{(\bar{v},X)}$, then by definition of f^C we have $Z' = Z$, and by Lemma 43 we get that B' is not an (X, Z)-bs in G_X. Finally, if $Z \in \mathcal{L}_2^{(*v,X)}$, then by definition of f^C we have $Z' \in \{Z, Z \cup \{v\}\}$, and again by Lemma 43 we get that B' is not an (X, Z)-bs in G_X. ◻

4.5 Putting pieces together

Let us now assume once again that the input graph G and the nice tree decomposition $\mathcal{D}$ of G are provided, and let us describe a recursive algorithm $\mathcal{A}$ that solves problem Π. We distinguish several cases as follows. In each case, namely join, introduce, or forget, we use the notations introduced in the corresponding lemma, namely Lemma 32, Lemma 38, or Lemma 45, respectively, and define Algorithm $\mathcal{A}$ as follows.

▶ **Definition 46.** *Suppose we are given an instance $I = (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S}) \in \mathcal{E}$ of problem Π such that X is a join node with children $X^L = X^R = X$. For each collection $\mathcal{P} = \{\mathcal{L}_1^A, \mathcal{L}_1^B, \mathcal{L}_1^C, \mathcal{L}_2^A, \mathcal{L}_2^B, \mathcal{L}_2^C\}$ such that*

- $\mathcal{L}_1 = \mathcal{L}_1^A \uplus \mathcal{L}_1^B \uplus \mathcal{L}_1^C$ *and*
- $\mathcal{L}_2 = \mathcal{L}_2^A \uplus \mathcal{L}_2^B \uplus \mathcal{L}_2^C$,

we denote by $I^L(\mathcal{P}) = (X^L, B_0, \mathcal{L}_1^L, \mathcal{L}_2^L, f^L, \mathcal{S}^L)$ and $I^R(\mathcal{P}) = (X^R, B_0, \mathcal{L}_1^R, \mathcal{L}_2^R, f^R, \mathcal{S}^R)$ as defined by Lemma 32.

In the join case, Algorithm $\mathcal{A}$ enumerates all such collections, and returns $\mathcal{A}(I^L(\mathcal{P})) \cup \mathcal{A}(I^R(\mathcal{P}))$, where $\mathcal{P}$ maximizes $|A(I^L(\mathcal{P}))| + |A(I^R(\mathcal{P}))|$.

▶ **Definition 47.** *Suppose we are given an instance $I = (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S}) \in \mathcal{E}$ of problem Π such that X is an introduce node with child $X^C = X \setminus \{v\}$. For each collection $\mathcal{P} = \{\mathcal{L}_2^A, \mathcal{L}_2^B\}$ of $\mathcal{L}_2$ as required in Case 1 of Lemma 38, we denote by $I^1(\mathcal{P})$ the instance defined in Case 1 of Lemma 38, and we denote by I^2 the instance defined in Case 2 of Lemma 38.*

In the introduce case, if $v \in B_0$ then Algorithm $\mathcal{A}$ enumerates all such collections and returns $\{v\} \cup \mathcal{A}(I^1(\mathcal{P}))$, where $\mathcal{P}$ maximizes $|\mathcal{A}(I^1(\mathcal{P}))|$, and if $v \notin B_0$ then Algorithm $\mathcal{A}$ returns $\mathcal{A}(I^2)$.

▶ **Definition 48.** *Suppose we are given an instance $I = (X, B_0, \mathcal{L}_1, \mathcal{L}_2, f, \mathcal{S}) \in \mathcal{E}$ of problem Π such that X is a forget node with child $X^C = X \cup \{v\}$. For each $Z^* \in \mathcal{L}_1$ as required in Case 1 of Lemma 45, we denote by $I^1(Z^*)$ the instance defined in Case 1 of Lemma 45, and for each partition $\mathcal{P} = \{\mathcal{S}^A, \mathcal{S}^B\}$ of $\mathcal{S}^{(*v,X)}$ and function f_2^C as required in Case 2 of Lemma 45, we denote by $I^2(\mathcal{P}, f_2^C)$ the instance defined in Case 2 of Lemma 45.*

In the forget case, Algorithm $\mathcal{A}$ enumerates all sets $Z^* \in \mathcal{L}_1$ as required in Case 1, and computes $B^1 = \mathcal{A}(I^1(Z^*))$ where Z^* maximizes $|I^1(Z^*)|$. Then, Algorithm $\mathcal{A}$ enumerates all sets $\mathcal{S}^A, \mathcal{S}^B$, and functions f_2^C as required in Case 2, and computes $B^2 = \mathcal{A}(I^2(\mathcal{P}, f_2^C))$ where $\mathcal{P}, f_2^C$ maximizes $|I^2(\mathcal{P}, f_2^C)|$. Finally, Algorithm $\mathcal{A}$ returns the largest solution among all B^1 and B^2.

In any of the three cases (join, forget, introduce), $\mathcal{A}$ returns $\max_{x \in E} \mathcal{A}(x)$ for some appropriate set E, and we point out that it may be the case that $E = \emptyset$, when none of the enumerated parameters respect the required conditions of Lemma 32, Lemma 38, and Lemma 45, and in this case Algorithm $\mathcal{A}$ returns $-\infty$ instead of a solution. Concerning the base case, we can always assume that the underlying tree decomposition has leaves where

$X = \emptyset$. On such a leaf, $\emptyset$ is the only candidate solution, and thus $\mathcal{A}$ returns $\emptyset$ if it is a valid solution, or $-\infty$ otherwise.

▶ **Lemma 49.** *$\mathcal{A}$ solves Π optimally: for every instance $I \in \mathcal{E}$, if I is feasible then $A(I)$ returns an optimal solution. Otherwise, it returns $-\infty$.*

Proof: The proof is by induction on the number of remaining bags in G_X in the provided nice tree decomposition $\mathcal{D}$. Let B be the solution returned by $\mathcal{A}(I)$, and let B^* be an optimal solution of I. We distinguish the different types of nodes in the nice tree decomposition $\mathcal{D}$ of G. In the three types of nodes, if I is not feasible, then by Lemma 32, Lemma 38, and Lemma 45, and by the inductive hypothesis, any of the recursive calls will output $-\infty$, and thus $\mathcal{A}(I)$ will return $-\infty$ as well. We suppose now that $\mathcal{I}$ is feasible, and let $B = \mathcal{A}(I)$ and let B^* an optimal solution.

Join node. By Lemma 32, there exists a collection $\mathcal{P}^*$ as defined in Definition 46 and sets B^{*L}, B^{*R} such that $B^{*L} \vdash I^L(\mathcal{P}^*)$ and $B^{*R} \vdash I^R(\mathcal{P}^*)$. Let $\mathcal{P}$ be the collection chosen by $\mathcal{A}$. We have $|B| = |A(I^L(\mathcal{P}))| + |A(I^L(\mathcal{P}))| - |B_0| \geq |A(I^L(\mathcal{P}^*))| + |A(I^L(\mathcal{P}^*))| - |B_0| \geq |B^{*L}| + |B^{*R}| - |B_0| = |B^*|$. Moreover, by Lemma 32, $B \vdash I$.

Introduce node. If $v \in B^*$, then according to Case 1 of Lemma 38 there exists a collection $\mathcal{P}^*$ as defined in Definition 47 such that $B^* \setminus \{v\} \vdash I^1(\mathcal{P}^*)$. Let $\mathcal{P}$ be the collection chosen by $\mathcal{A}$. As in this case we have $v \in B_0$, we have $|B| = 1 + |A(I^1(\mathcal{P}))| \geq 1 + |A(I^1(\mathcal{P}^*))| \geq 1 + |B^* \setminus \{v\}| = |B^*|$. Moreover, by Lemma 38, $B \vdash I$. If $v \notin B^*$, then according to Case 2 of Lemma 38, $B^* \vdash I^2$. As in this case we have $v \notin B_0$, we have $|B| = |A(I^2)| \geq |B^*|$. Moreover, according to Lemma 38, $B \vdash I$.

Forget node. If $v \in B^*$, then according to Case 1 of Lemma 45 there exist $Z^{**} \in \mathcal{L}_1$ such that $B^* \vdash I^1(Z^{**})$. Let Z^* be the element chosen by $\mathcal{A}$ for the first case, and $(\mathcal{P}, f_2^C)$ the elements chosen for the second case. We have $|B| \geq |A(I^1(Z^*))| \geq |A(I^1(Z^{**}))| \geq |B^*|$. If $v \notin B^*$, then according to Case 2 of Lemma 45 there exist $\mathcal{P}^*$ and f_2^{*C} such that $B^* \vdash I^2(\mathcal{P}^*, f_2^{*C})$. We have $|B| \geq |A(I^2(\mathcal{P}, f_2^C))| \geq |A(I^2(\mathcal{P}^*, f_2^{*C}))| \geq |B^*|$. ◻

▶ **Lemma 50.** *Algorithm $\mathcal{A}$ runs in time $\mathcal{O}^*(2^{\mathcal{O}(2^t)})$, where t is the width of the given nice tree decomposition of the input graph.*

Proof: The time complexity of Algorithm $\mathcal{A}$ is $\mathcal{O}^*(x_1 \cdot x_2)$, where $x_1 = |\mathcal{E}|$ is the number of possible inputs of Π and x_2 is the maximum time necessary to compute $\mathcal{A}(I)$ for each $I \in \mathcal{E}$. We denote $n = |V(G)|$. From the definitions of the corresponding objects, it can be routinely verified that $x_1 \leq n \cdot 2^t \cdot 2^t \cdot 2^{2^t} \cdot 2^{2^t} \cdot (2^t)^t \cdot 2^t \leq 2^{3t+2^{t+1}+t^2} = 2^{\mathcal{O}(2^t)}$.

Let us now bound x_2. To that end, let θ_1 be an upper bound on the number of enumerated subinstances made in any of the three cases (join, introduce, or forget) and let by θ_2 be an upper bound on the time complexity related to all operations like taking the minimum, and verifying that each enumerated subinstance verifies required properties (like for example, in Case 1 of Lemma 45). In the join case, the number of subinstances is at most $3^{|\mathcal{L}_1|} \cdot 3^{|\mathcal{L}_2|} \leq 3^{2^{t+1}}$ (to consider all $\mathcal{L}_1^A, \mathcal{L}_1^B, \mathcal{L}_1^C, \mathcal{L}_2^A, \mathcal{L}_2^B, \mathcal{L}_2^C$), and for each subinstance the complexity is polynomial in n. In the introduce case, the number of subinstances is $2^{2^t} + 1$ (to consider all $\mathcal{L}_2^A$), and for each subinstance the complexity is polynomial in n. In the forget case, the number of subinstances is at most $2^{2^t} + 2^{2^t} \cdot 2^{2^t}$ (to consider all $Z^* \in \mathcal{L}_1$ in Case 1 and all $\mathcal{S}^A$ and f^C in Case 2), and for each subinstance the complexity is in $\mathcal{O}^*(2^t)$ as in Case 1, for every $Z \in \mathcal{S}$ (resp. every $v' \in B_0$,) we must verify if (X, Z) (resp. $(X, f(v'))$) is not v-critical in G_X, and this verification can be done in time $\mathcal{O}^*(2^t)$ according to Lemma 40. All in all, we can choose $\theta_1 = 3^{2^{t+2}} = 2^{\mathcal{O}(2^t)}$, $\theta_2 = \mathcal{O}^*(2^t)$, and the lemma follows. ◻

As according to [6] we can determine whether $\mathsf{tw}(G) \leq t$ in time $\mathcal{O}^*(t^{\mathcal{O}(t^3)})$, and construct the corresponding (nice) tree decomposition of G in case of a positive answer, from Proposition 29 and Lemma 50 the following theorem is now immediate.

▶ **Theorem 51.** *The* MMBS/tw *problem is* FPT. *More precisely, it can be solved in time* $\mathcal{O}^*(2^{\mathcal{O}(2^{\mathsf{tw}(G)})})$.

5 Further research

We presented a number of negative and positive results for the MMBS and MMHS problems. Several interesting questions remain open. Concerning MMBS, even if it seems implausible that the problem could be expressed in monadic second-order logic, it would be nice to prove it formally. For that, one may try to use the framework introduced by van Bevern et al. [27]. While it is easy to see that, if we consider the combined parameter $\mathsf{tw} + \alpha$, then the MMBS problem can be expressed in monadic second-order logic, it is not clear that taking the combined parameter $\mathsf{tw} + \beta$ helps. Simplifying the dynamic programming algorithm behind Theorem 51 is also worth trying, in particular with the combined parameter $\mathsf{tw} + \beta$. Another direction is to either find more efficient FPT algorithms for parameters like feedback vertex set, treedepth and vertex cover, or to consider the parameterization by cliquewidth.

Taking into account the motivation to study the MMBS problem discussed in Section 1 in the context of the kernelization of Vertex Cover, it would be interesting to study the complexity of MMBS restricted to minor-free input graphs, such as planar graphs.

As for MMHS, we believe that the main challenge is trying to get a algorithm parameterized by $\alpha + \beta$ running faster than $\mathcal{O}^*(2^{\alpha\beta})$ (see Theorem 24). Let us consider the case $\alpha = 2$, corresponding to the MMVC problem. The parameterized complexity of MMVC has received some attention recently, with results concerning FPT algorithms in time $\mathcal{O}^*(2^{\beta})$ [20], and even in time $\mathcal{O}^*(c^{\beta})$ for $c < 1.54$ [8], FPT algorithms for structural parameterizations [28], and kernelization [1]. This motivates the problem of trying to improve the running time $\mathcal{O}^*(2^{\alpha\beta})$ of Theorem 24, for example by solving MMHS in time $\mathcal{O}^*(\alpha^{\beta})$, or even $\mathcal{O}^*(\alpha^{\mathcal{O}(\beta)})$. Recall that the algorithm of Proposition 13 runs in time $\mathcal{O}^*(\alpha^{\beta})$ for *fixed* α, and that it hides a term $|V(\mathcal{H})|^{f(\alpha)}$ for some function f.

Achieving a running time of $\mathcal{O}^*(\alpha^{\beta})$ might be typically done by guessing, at each step, only which vertex of a given hyperedge should be added to the solution. However, guessing only a vertex v and applying recursion on a remaining instance $(\mathcal{H}', \beta-1)$, where $\mathcal{H}'$ is defined by removing v and all hyperedges containing v, is not correct. Indeed, $(\mathcal{H}', \beta - 1)$ being a yes-instance, certified by a solution S', does not imply that $(\mathcal{H}, \beta)$ is also a yes-instance, as $S' \cup \{v\}$ may not be minimal anymore. Thus, we believe that, in order to solve MMHS in time $\mathcal{O}^*(\alpha^{\beta})$ or even $\mathcal{O}^*(\alpha^{\mathcal{O}(\beta)})$, a significantly new approach should be devised.

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