Arc-disjoint in- and out-branchings in digraphs of independence number at most 2

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Arc-disjoint in- and out-branchings in digraphs of independence number at most 2*

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Abstract

We prove that every digraph of independence number at most 2 and arc-connectivity at least 2 has an out-branching B^+ and an in-branching B^- which are arc-disjoint (we call such branchings a **good pair**). This is best possible in terms of the arc-connectivity as there are infinitely many strong digraphs with independence number 2 and arbitrarily high minimum in-and out-degrees that have no good pair. The result settles a conjecture by Thomassen for digraphs of independence number 2. We prove that every digraph on at most 6 vertices and arc-connectivity at least 2 has a good pair and give an example of a 2-arc-strong digraph D on 10 vertices with independence number 4 that has no good pair. We also show that there are infinitely many digraphs with independence number 7 and arc-connectivity 2 that have no good pair. Finally we pose a number of open problems.
Keywords: Arc-disjoint branchings, out-branching, in-branching, digraphs of independence number 2, arc-connectivity.

1 Introduction

It is a well-known result due to Nash-Williams and Tutte [12, 14] that every $2k$-edge-connected graph has a set of k edge-disjoint spanning trees. For digraphs, there are many possible analogues of a spanning tree. The two most natural ones are out-branchings and in-branchings. An **out-branching** (**in-branching**) of a digraph D is a spanning tree in the underlying graph of D whose edges are oriented in D such that every vertex except one, called the root, has in-degree (out-degree) one. Edmonds [8] characterized digraphs with k arc-disjoint out-branchings with prescribed roots. Lovász [10] gave an algorithmic proof of Edmonds' result which implies that one can check in polynomial time, for each fixed natural number k, whether a given digraph has a collection of k arc-disjoint out-branchings (roots not specified).

No good characterization is known for digraphs having an out-branching and an in-branching which are arc-disjoint and very likely none exists, due to the following result.

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Theorem 1 (Thomassen (see [1]))**.** *It is NP-complete to decide whether a given digraph D has an out-branching and an in-branching both rooted at the same vertex such that these are arc-disjoint.*

This implies that it is also NP-complete to decide if a digraph has any out-branching which is arc-disjoint from some in-branching, see Theorem 5. The same conclusion holds already for 2-regular digraphs [6].

Thomassen also conjectured that every digraph of sufficiently high arc-connectivity should have such a pair of branchings, where the arc connectivity is the minimum out-degree of a proper subset of the vertex set of the considered digraph. His conjecture was for branchings with the same root, but as we show in Proposition 4, the conjecture is equivalent to the following.

Conjecture 2 (Thomassen [13])**.** There is a constant C, such that every digraph with arc-connectivity at least C has an out-branching and an in-branching which are arc-disjoint.

Conjecture 2 has been verified for semicomplete digraphs [1] and for locally semicomplete digraphs [5]. In both cases arc-connectivity 2 suffices. For general digraphs the conjecture is wide open and as far as we know it is not known whether already $C = 3$ would suffice in Conjecture 2 (Figure 10 below shows that $C = 2$ is not sufficient).

Our main result is the following which verifies Thomassen's conjecture for digraphs of independence number 2.

Theorem 3. *If D is a digraph with $\alpha(D) \leq 2 \leq \lambda(D)$, then D has a good pair.*

The paper is organized as follows. In Section 2 we provide som notation and preliminary results. In particular, we show in Proposition 10 that there is no lower bound on the minimum in- and out-degree that suffices to guarantee that a strongly connected digraph with independence number 2 has a good pair. The proof of Theorem 3, which we give in Section 6, builds on several partial results which we establish in Sections 3 to 5. In Section 3 we characterize semicomplete digraphs with good pairs. Then in Section 4 we give some sufficient conditions for a digraph on at most 6 vertices to have a good pair. In Section 5 we consider co-bipartite digraphs which are digraphs whose vertex set can be covered by two semicomplete subdigraphs. We prove that every 2-arc-strong co-bipartite digraph has a good pair hence allowing us to focus on digraphs which are not co-bipartite in Section 6 where we prove of Theorem 3.

In Section 7 we provide an example to show that arc-connectivity 2 is not sufficient to guarantee that a digraph with independence number 2 has an out-branching rooted at a prescribed vertex s which is arc-disjoint from an in-branching rooted at a prescribed vertex t for every choice of vertices s, t. We also show that there are infinitely many digraphs with independence number 3 and arc-connectivity 2 which do not have arc-disjoint out- and in-branchings, B_s^+, B_t^-, rooted at some given s and t, respectively. Using this we construct an infinite family of digraphs with independence number 7 and arc-connectivity 2 which have no out-branching which is arc-disjoint from some in-branching. We also show that every 2-arc-strong digraph on at most 6 vertices has an out-branching which is arc-disjoint from some in-branching. Finally, we pose a number of open problems. Finally in Section 8 we list some open problems and conjectures.

2 Notation and preliminaries

Notation not given below follows [3, 4]. The digraphs in this paper have no loops and no multiple arcs. Let $D = (V, A)$ be a digraph. Let $X, Y \subset V$ be two sets of vertices. If D contains the arc xy for every choice of $x \in X$ and $y \in Y$, then we write $X \rightarrow Y$. If moreover, there is no arc with tail in Y and head in X, then we write $X \mapsto Y$.

If D' is a subdigraph of D and uv is an arc of D, then we denote by $D' + uv$ the digraph with vertex set $V(D') \cup \{u, v\}$ and arc set $A(D') \cup \{uv\}$.

For a non-empty subset $X \subset V$ we denote by $d_D^+(X)$ (resp. $d_D^-(X)$) the number of arcs with tail (resp. head) in X and head (resp. tail) in $V - X$. We call $d_D^+(X)$ (resp. $d_D^-(X)$) the **out-degree** (resp. **in-degree**) of the set X. Note that X may be just a vertex. We will drop the subscript when the digraph is clear from the context. We denote by $\delta^0(D)$ the minimum over all in- and out-degrees of vertices of D. This is also called the minimum **semidegree** of a vertex in D. The **arc-connectivity** of D, denoted by $\lambda(D)$, is the minimum out-degree of a proper subset of V. A digraph is strongly connected (or just **strong**) if $\lambda(D) \geq 1$.

In- and out-branchings were defined above. We denote by B_s^+ (respectively B_t^-) an out-branching rooted at s (respectively an in-branching rooted at t). We also use B^+ or O (resp. B^- or I) to denote an out-branching (resp. in-branching) with no root specified.

Proposition 4. *Let R be a natural number. If there exists a digraph H with $\lambda(H) = R$ which has two (possibly equal) vertices s and t such that H has no pair of arc-disjoint branchings B_s^+, B_t^-, then there exists a digraph U with $\lambda(U) = R$ which has no out-branching which is arc-disjoint from some in-branching.*

Proof. Let H be given as above. If $s = t$, then let $H' = H$ and otherwise we obtain H' by adding a copy X of the complete digraph on R vertices, all possible arcs from $V(X)$ to s and all possible arcs from t to $V(X)$. It is easy to check that $\lambda(H') = R$ and that H' has no pair of arc-disjoint branchings B_x^+, B_x^- where $x \in X$. Now if $s = t$ take $x = s = t$ and if $s \neq t$ fix one vertex $x \in X$. Let U be the digraph that we obtain from three disjoint copies of H' by identifying the copies of x in these. Then $\lambda(U) = \lambda(H') = R$ and U has no pair of arc-disjoint branchings B_u^+, B_v^- for any choice of vertices u, v. This follows from the fact that U could only have such branchings if one copy of H' would have arc-disjoint branchings B_x^+, B_x^-. $\square$

The following is an easy consequence of Theorem 1 and the construction above.

Theorem 5. *It is NP-complete to decide if a given digraph has an out-branching and in-branching which are arc-disjoint.*

A vertex v of a digraph $D = (V, A)$ is an **in-generator** (resp. **out-generator**) if v can be reached from (resp. can reach) every other vertex in V by a directed path. Thus a vertex v is an in-generator (resp. out-generator) if and only if v is the root of some in-branching B_v^- (resp. out-branching B_v^+) of D. The set of in-generators (resp. out-generators) of a digraph is denoted by $\mathrm{In}(D)$ (resp. $\mathrm{Out}(D)$). A **strong component** of a digraph D is a set of vertices inducing a strong digraph and maximal by inclusion for this property. A strong component X is **initial** (resp. **terminal**) if $V(X)$ has no in-coming (resp. out-going) arcs in D. The in-generators (resp. out-generators) of D are precisely the vertices of the intial (resp. terminal) components of D.

If $X \subset V$ we denote by $D\langle X\rangle$ the subdigraph of D **induced** by X, that is, the digraph whose vertex set is X and whose arc set consists of those arcs from A that have both end-vertices in X.

A digraph is **semicomplete** if it has no pair of non-adjacent vertices. A **tournament** is a semicomplete digraph with no directed cycle of length 2. We need the following results on semicomplete digraphs. For a survey on results for semicomplete digraphs see Chapter 2 in [3]. The following is an easy consequence of the definition of strong connectivity.

Lemma 6. *Let D be semicomplete digraph. Then the induced subdigraphs $D\langle In(D)\rangle$ and $D\langle Out(D)\rangle$ are strong.*

A digraph $D = (V, A)$ is **vertex-pancyclic** if for every $3 \leq l \leq |V|$ and every vertex $v \in V$ there exists a directed cycle of D of length l containing v.

Theorem 7 (Moon [11])**.** *Every strong semicomplete digraph on at least 3 vertices is vertex-pancyclic. In particular, every strong semicomplete digraph on at least 2 vertices has a hamiltonian cycle.*

An **independent set** in a digraph $D = (V, A)$ is a set $X \subseteq V$ such that $D\langle X\rangle$ has no arcs. We denote by $\alpha(D)$ the maximum size of an independent set in D.

Theorem 8 (Chen-Manalastras [7])**.** *Let D be a strong digraph with $\alpha(D) = 2$. Then either D has a directed hamiltonian cycle or its vertices can be covered by two directed cycles C_1, C_2 such that these are either vertex disjoint or they intersect in a subpath of both. In particular D has a directed hamiltonian path.*

By a **clique** in a digraph we mean an induced subdigraph which is semicomplete. The following is a well known consequence of a result in Ramsey theory.

Theorem 9. *Every digraph on at least* 9 *vertices contains either an independent set of size at least* 3 *or a clique of size* 4*.*

A **good pair** (in D), is a pair (I, O) such that I is an in-branching of D, O is an out-branching of D, and I and O are arc-disjoint. A **good r-pair** (in D), is a good pair (I, O) such that r is the root of I. A **good (r, q)-pair** (in D), is a good pair (I, O) such that r is the root of I and q is the root of O.

A digraph is **co-bipartite** if its underlying graph is the complement of a bipartite graph. In other words, its vertex set can be partitioned in two sets V_1, V_2 such that $D\langle V_1\rangle$ and $D\langle V_2\rangle$ are semicomplete digraphs.

Proposition 10. *For every natural number k, there are infinitely many strong co-bipartite digraphs with minimum semidegree at least k and no good pair.*

Proof. Let T_1, T_2 be strongly connected tournaments with $\delta^0(T_i) \geq k$ for $i = 1, 2$ and let T be obtained from these by adding a new vertex v and all possible arcs from v to $V(T_1)$, all possible arcs from $V(T_2)$ to v and all possible arcs from $V(T_2)$ to $V(T_1)$, except one arc xy with goes from $V(T_1)$ to $V(T_2)$. The resulting digraph T is clearly strong and does not have an out-branching and an in-branching, both rooted at v, which are arc-disjoint (the arc xy must belong to both branchings). Now let H be the digraph that we obtain from two copies H', H'' of H by adding a 2-cycle between the two copies v', v'' of v. This digraph is clearly co-bipartite. Suppose H has a good pair. Then, w.l.o.g., the root of the out-branching belongs to H' and then also the root of the in-branching must belong to H' (as the arcs $v'v''$ and $v''v'$ are the only arcs between the two copies of H). But that means that H'' has an in-branching rooted at v'' which is arc-disjoint from an out-branching rooted at v'', a contradiction. □

3 Good pairs in semicomplete digraphs

We first consider semicomplete digraphs and derive some easy results that will be used later.

Lemma 11. *Let D be a non-strong semicomplete digraph of order at least* 4 *and let $r \in \mathrm{In}(D)$ and $q \in \mathrm{Out}(D)$ be arbitrary. Then D has a good (r, q)-pair, (I, O), in D.*

Proof. Let W_1 respectively W_2 denote $D\langle \mathrm{Out}(D)\rangle$ respectively $D\langle \mathrm{In}(D)\rangle$.

First consider the case when $|\mathrm{Out}(D)| \geq 2$, in which case we can let $U = (u_1, u_2, \ldots, u_a, u_1)$ be a hamiltonian cycle in W_1 ($a = |\mathrm{Out}(D)|$) such that $u_1 = q$, which exists by Theorem 7. Let I' be any in-branching in $D \setminus W_1$ with root r, which exists as $\mathrm{In}(D) = \mathrm{In}(D \setminus W_1)$. We now construct I from I' by adding the arc $u_a u_1$ and every arc from $\{u_1, u_2, \ldots, u_{a-1}\}$ to r. We construct O by taking the path $u_1 u_2 \ldots u_a$ and adding every arc from u_a to $V(D) \setminus \mathrm{Out}(D)$. This gives us the desired good (r, q)-pair, (I, O), in D.

We may therefore assume that $|\mathrm{Out}(D)| = 1$ and analogously that $|\mathrm{In}(D)| = 1$. This implies that $\mathrm{Out}(D) = \{q\}$ and $\mathrm{In}(D) = \{r\}$. As the order of D is at least 4, there exists an arc uv in $D - \{q, r\}$. The following out-branching, O, and in-branching, I, form the desired good (r, q)-pair, (I, O), in D.

$$A(I) = \{qv, vr, ur\} \cup \{zr \mid z \in V(D) \setminus \{r, q, u, v\}\}$$

$$A(O) = \{qu, uv, qr\} \cup \{qz \mid z \in V(D) \setminus \{r, q, u, v\}\}$$

□

Lemma 12. *Let D be a semicomplete digraph and r be a vertex in $\mathrm{In}(D)$. If there is a subdigraph D' of D of order at least* 2 *having a good r-pair, (I', O'), then D has a good r-pair, (I, O).*

Proof. Let (I', O') be a good r-pair of D'. Assume that $V(D') \neq V(D)$. Since r is an in-generator in D, there is a vertex $y \in V(D) \setminus V(D')$ and and arc yx with $x \in V(D')$. If $y \to V(D')$, then let r' be the root of O' and $v \in V(D') \setminus r'$. Set $I = I' + yv$ and $O = O' + yr'$. Then (I, O) is a good r-pair of $D\langle V(D') \cup \{y\}\rangle$. If $y \not\to V(D')$, then there exist two vertices $z_1, z_2 \in V(D')$ such that $z_2 y$ and yz_1 are arcs of D. Set $I = I' + yz_1$ and $O = O' + z_2 y$. Then (I, O) is a good r-pair of $D\langle V(D') \cup \{y\}\rangle$.

We can apply this process iteratively until we obtain a good r-pair of D. □

A 4**-exception** is a pair (D, a) such that D has 4 vertices and contains the strong tournament of order 4 depicted in Figure 1 (with plain arcs) and possibly one or both arcs in $\{dc, cb\}$ (shown as dotted arcs).

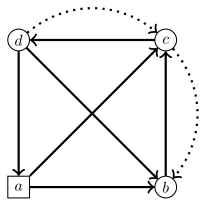

Figure 1: The 4-exceptions (D, a).

Proposition 13. *Let D be a semicomplete digraph of order 4 and let r be a vertex of* $\text{In}(D)$*. Then D has a good r-pair unless (D, r) is a 4-exception.*

Proof. One easily sees that D contains a spanning tournament T such that $r \in \text{In}(T)$. There are only four tournaments of order 4, ST_4 the unique strong tournament of order 4 and the three non-strong tournaments depicted in Figure 2. For each of these tournaments, by symmetry, we may assume that r is the white vertex and a good r-pair is given in Figure 2.

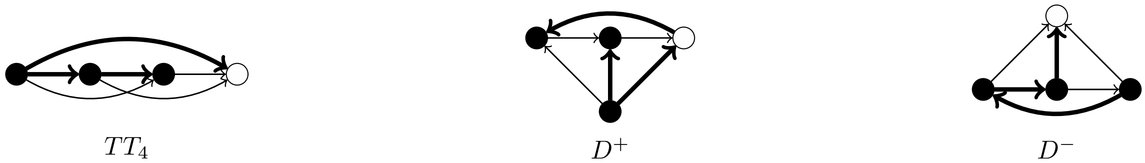

Figure 2: The non-strong tournaments of order 4 and a good r-pair when r is the white vertex. The arcs of the in-branching are thin and the arcs of the out-branching are thick.

Henceforth, we may assume that T is ST_4. If $r \in \{b, c, d\}$, then there is a good r-pair as shown in Figure 3.

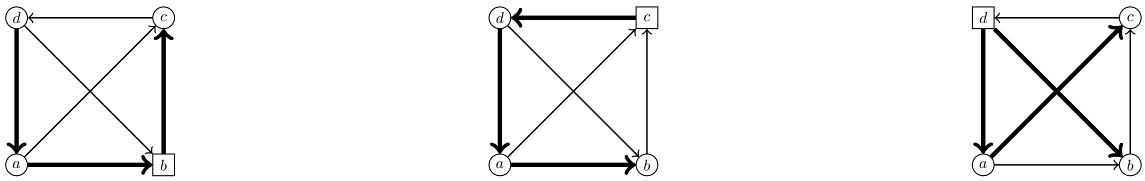

Figure 3: Good r-pairs in ST_4 for $r \in \{b, c, d\}$. The arcs of the in-branching are thin and the arcs of the out-branching are thick.

Henceforth we may assume that $r = a$. If D contains one of the arcs ba, ad, ca, bd, then there is a good r-pair as shown in Figure 4.

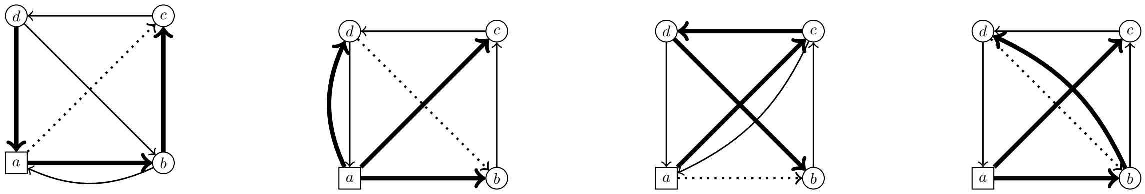

Figure 4: Good a-pairs in digraphs containing ST_4 and an arc in $\{ba, ad, ca, bd\}$. The arcs of the in-branching are thin and the arcs of the out-branching are thick.

If not, then (D, a) is a 4-exception. In such a case, there is no good a-pair. Indeed if there were one, then the in-branching must contain the arcs da and cd, and $D \setminus \{da, cd\}$ has no out-branching because it has two sources (namely a and d). □

An **exception** is a pair (D, r) where D is a semicomplete digraph, r is a vertex of D, such that $N^-(r) = \{y\}$ and $d^-(y) = 1$.

Theorem 14. *Let D be a semicomplete digraph of order at least 4 and let $r \in \mathrm{In}(D)$. There is a good r-pair if and only if (D, r) is not an exception.*

Proof. If (D, r) is an exception, then let $N^-(r) = \{y\}$ and $N^-(y) = \{z\}$. Note that any in-branching, I, with root r, must contain the arcs yr and zy. However, in $D \setminus \{yr, zy\}$ we note that both r and y have in-degree zero, and therefore there is no out-branching in $D \setminus \{yr, zy\}$, which implies that there is no good r-pair in D.

So now assume that (D, r) is not an exception. If D is non-strong then we are done by Lemma 11, so we may assume that D is strongly connected. By Theorem 7, r is in a directed 3-cycle $zyrz$. If there exists $t \in V(D) \setminus \{z, y, r\}$ such that $r \in \mathrm{In}(D\langle\{z, y, r, t\}\rangle)$ and $(D\langle\{z, y, r, t\}\rangle, r)$ is not a 4-exception, then, by Lemma 12, D contains a good r-pair. Henceforth, we may assume that, for all $t \in V(D) \setminus \{z, y, r\}$, either $r \notin \mathrm{In}(D\langle\{z, y, r, t\}\rangle)$ or $(D\langle\{z, y, r, t\}\rangle, r)$ is a 4-exception. In both cases, $r \mapsto t$ and $y \mapsto t$. Hence $r \mapsto V(D) \setminus \{y, r\}$, and $y \mapsto V(D) \setminus \{r, y, z\}$. If $r \to y$, then $D' = D \setminus \{yr\}$, is a semicomplete digraph with $\mathrm{In}(D') = \{r\}$, and by Lemma 11, D' has a good r-pair, which is also a good r-pair in D. If not, then $N^-(r) = \{y\}$ and $N^-(y) = \{z\}$, a contradiction to (D, r) not being an exception. □

Corollary 15. *Every semicomplete digraph D of order at least 4 has a good pair.*

Proof. If D is non-strong the corollary follows from Lemma 11, so assume that D is strongly connected. As D has order at least 4, we note that there exists a vertex $r' \in V(D)$ such that $d^-(r') \geq 2$. This implies that (D, r') is not an exception and therefore there exists a good r'-pair (I, O). □

In particular, notice that if D is semicomplete and $\delta^0(D) \geq 2$, then either D is of order at least 4 or D is a complete digraph on 3 vertices and in both cases, it admits a pair of arc-disjoint in- and out-branchings.

4 Good pairs in small digraphs

Lemma 16. *Let D be a digraph and $X \subset V(D)$ be a set such that every vertex of X has both an in-neighbour and an out-neighbour in $V \setminus X$. If $D \setminus X$ has a good pair then D has a good pair.*

Proof. Let (I, O) be a good pair of $D \setminus X$. By assumption, every $x \in X$ has an out-neighbour y_x in $V(D) \setminus X$ and an in-neighbour w_x in $V(D) \setminus X$. Then $(I + \{xy_x \mid x \in X\}, O + \{w_x x \mid x \in X\})$ is a good pair for D. □

Proposition 17. *Every digraph on 3 vertices with at least 4 arcs has a good pair.*

Proof. Let D be a digraph on 3 vertices a, b, c and with at least 4 arcs. By symmetry, and without loss of generality, D has a 2-cycle (a, b, a) and bc is an arc. Then the path $P = (a, b, c)$ is both an in- and an out-branching. But $Q = D \setminus A(P)$ is a path of length 2, which is necessarily an out- or an in-branching. Hence either (P, Q) or (Q, P) is a good pair of D. □

Let E_4 be the digraph depicted in Figure 5.

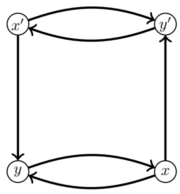

Figure 5: The digraph E_4.

Proposition 18. *Let D be a digraph of order 4 with at least 6 arcs and with $\delta^0(D) \geq 1$. Then D has a good pair if and only if $D \neq E_4$.*

Proof. Observe that D has at least as many directed 2-cycles as it has pairs of non-adjacent vertices.

It is easy to check that E_4 has no good pair. Assume that $D \neq E_4$ and has no good pair. Then, by Corollary 15, D is not semicomplete. Hence it has at least one pair of non-adjacent vertices and thus at least one directed 2-cycle C. By Proposition 17 and Lemma 16, D has no subdigraph of order 3 with at least 4 arcs. In particular, every vertex x in $V(D) \setminus V(C)$ is adjacent to at most one vertex of C. Hence D contains at least two pairs of non-adjacent vertices and thus at least two directed cycles. Furthermore, no two directed cycles can intersect, for otherwise their union is a digraph of order 3 with four arcs. Hence D has exactly two directed 2-cycles, C and C', and there are two pairs of non-adjacent vertices forming a matching between the vertices of C and C'. Since $D \neq E_4$, it must be the digraph F_4 depicted in Figure 6. But, as shown in Figure 6, F_4 has a good pair. □

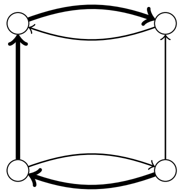

Figure 6: Digraph F_4 and one of its good pairs. The arcs of the in-branching are thin and the arcs of the out-branching thick.

Proposition 19. *Every digraph D with $\delta^0(D) \geq 2$ and order at most 5 has a good pair.*

Proof. Since $\delta^0(D) \geq 2$, the order n of D is at least 3.

If $n = 3$, then D is the complete digraph on three vertices (i.e. with all possible arcs), which contains a good pair.

If $n = 4$, then D has at least 8 arcs, and so at least two directed 2-cycles. A directed 2-cycle has a good pair, so by Lemma 16, D has a good pair.

If $n = 5$, then D has at least 10 arcs. Hence D has at least as many directed 2-cycles as it has pairs of non-adjacent vertices. If D contains a semicomplete digraph D' on 4 vertices, then, by Corollary 15, D' has a good pair, and so by Lemma 16, D has a good pair. Henceforth, we may assume that D contains no semicomplete digraph of order 4. Thus D has at least two pairs of non-adjacent vertices and thus at least two directed 2-cycles.

If D contains a subdigraph on 3 vertices with 4 arcs, then this digraph has a good pair by Proposition 17, and so D has a good pair by Lemma 16. Henceforth we may assume that D contains no subdigraph on 3 vertices with 4 arcs. But this is impossible, indeed if this was the case, then all directed 2-cycles are vertex disjoint, so there are at most two of them, and each directed 2-cycle is incident to at least three non-edges. This contradicts the fact that the number of pairs of non-adjacent vertices is no greater than the number of directed 2-cycles. $\square$

Lemma 20. *If $D = (V, A)$ is a digraph on n vertices with $\lambda(D) = 2$ that contains a subdigraph on $n-3$ vertices with a good pair, then D has a good pair. In particular, if D has 6 vertices and contains a subdigraph on 3 vertices which has at least 4 arcs then D has a good pair.*

Proof. First, notice that the second part of the statement follows from Proposition 17. Now, let $X \subset V$ be a subset of size $n-3$ such that $D\langle X\rangle$ has a good pair and let $V \setminus X = \{a, b, c\}$. If some vertex $v \in \{a, b, c\}$ has both an in-neighbour and an out-neighbour in X, then $D\langle X + v\rangle$ has a good pair and then the claim follows from Lemma 16, so we can assume there is no such vertex v. Then there cannot exist two vertices of $V \setminus X$ with an in-neighbour in X and two vertices of $V \setminus X$ with an out-neighbour in X. As $\lambda(D) \geq 2$, we may assume w.l.o.g. that a has two in-neighbours in $x_1, x_2 \in X$ and also that c has an out-neighbour x_1' in X. This implies that ab, ac are arcs of D and also bc as c has no in-neighbour in X. Suppose first that b has no in-neighbour in X, then cb is also an arc and now we can extend the good pair of $D\langle X\rangle$ by adding the arcs x_1a, ac, cb to the out-branching of $D\langle X\rangle$ and adding the arcs ab, bc, cx_1' to the in-branching of $D\langle X\rangle$. In the case when b has an in-neighbour x in X, we can extend the good pair of $D\langle X\rangle$ by adding the arcs x_1a, xb, ac to the out-branching of $D\langle X\rangle$ and adding the arcs ab, bc, cx_1' to the in-branching of $D\langle X\rangle$ $\square$

5 Good pairs in co-bipartite digraphs

Theorem 21. *Let D be a co-bipartite digraph with $\lambda(D) \geq 2$. Then D has a good pair.*

Proof. Let D be a co-bipartite digraph with vertex partition (V_1, V_2), that is $D_i = D\langle V_i\rangle$ is a semicomplete digraph for $i = 1, 2$. Without loss of generality, we may assume $|V_1| \leq |V_2|$. If $|V(D)| \leq 5$, then we have the result by Proposition 19. Therefore we may assume $|V(D)| \geq 6$. We distinguish several cases.

Case 1: $|V_1| \geq 4$. Let a_1a_2 be an arc from $\mathrm{In}(D_1)$ to V_2 in D, which exists as D is strongly connected and there is no arc from $\mathrm{In}(D_1)$ to $V_1 \setminus \mathrm{In}(D_1)$.

First assume that (D_1, a_1) is an exception and denote by y_1 the unique in-neighbour of a_1 in D_1 and by z_1 the unique in-neighbour of y_1 in D. By Corollary 15, as D_2 contains at least four vertices, it admits a good pair (I_2, O_2). Moreover, let I_1 be an in-branching of D_1 rooted at a_1, and O_1 be the out-branching of $D_1 \setminus y_1$ containing all the arcs leaving a_1. Now, as a_1 and y_1 have in-degree at least 2 in D they respectively have an in-neighbour a_1' and y_1' in V_2. Then $(I_2 + a_1a_2 + I_1, O_2 + a_1'a_1 + y_1'y_1 + O_1)$ is a good pair of D.

Assume now that (D_1, a_1) is not an exception. By Theorem 14, D_1 admits a good a_1-pair (I_1, O_1). We shall find a similar pair for D_2. As $\lambda(D) \geq 2$, $D \setminus a_1a_2$ is strong and so, there exists an arc b_1b_2 of $D \setminus a_1a_2$ from V_1 to $\mathrm{Out}(D_2)$. Consider the digraph $\tilde{D}$ obtained from D by reversing all its arcs, and set $\tilde{D_2} = \tilde{D}\langle V_2\rangle$. As $\mathrm{In}(\tilde{D_2}) = \mathrm{Out}(D_2)$, b_2 is a vertex of $\mathrm{In}(\tilde{D_2})$. If $(\tilde{D_2}, b_2)$ is an exception, then we conclude as previously that $\tilde{D}$ has a good pair. Thus, D has also a good pair. Otherwise, if $(\tilde{D_2}, b_2)$ is not an exception, by Theorem 14, $\tilde{D_2}$ admits a good b_2-pair $(\tilde{O_2}, \tilde{I_2})$. It means that D_2 admits a good pair (I_2, O_2) such that b_2 is the root of out-branching O_2. In this case, $(I_2 + a_1a_2 + I_1, O_1 + b_1b_2 + O_2)$ is a good pair of D.

Case 2: $|V_1| \leq 2$. Then $|V_2| \geq 4$ since $|V(D)| \geq 6$. Hence, by Corollary 15, D_2 has a good pair. Thus, by Lemma 16, D has also a good pair.

Case 3: $|V_1| = 3$. If D_2 admits a good pair, then we conclude by Lemma 20. So we can assume that D_2 has no good pair. By Corollary 15 we have $|V_2| = 3$. If D_1 has a good pair, then we apply Lemma 20. Therefore we may assume that D_1 has also no good pair. By Proposition 17, D_1 and D_2 are tournaments.

This implies that each vertex of V_i is incident to at least two arcs whose other end-vertex is in V_{3-i} for all $i \in [2]$.

If D contains a semicomplete subdigraph D' of order 4, then by Corollary 15, D' has a good pair, and so by Lemma 16, D has a good pair. Thus we may assume that D has no such subdigraph. In particular each vertex of V_i is non-adjacent to at least one vertex in V_{3-i} for all $i \in [2]$. Suppose that some vertex $v_1 \in V_1$ forms a 2-cycle with a vertex $v_2 \in V_2$. Then we can assume, by Lemma 20, that v_i is non-adjacent to the two other vertices of V_{3-i} for $i = 1, 2$. By the remark above $D \setminus \{v_1, v_2\}$ is not semicomplete so it contains two non-adjacent vertices which are respectively in V_1 and V_2. As these vertices have in- and out-degree at least 2, we conclude that each vertex of V_i forms a 2-cycle with one vertex of V_{3-i} for $i = 1, 2$. Furthermore, we can assume that none of these 2-cycles have a vertex in common, otherwise we conclude with Lemma 20. Now we see that D is one of the two digraphs in Figure 7 in which we show a good pair for D.

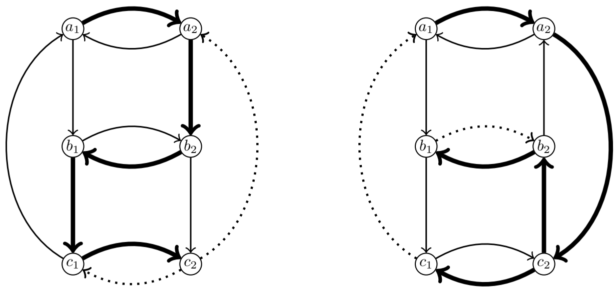

Figure 7: The two possible digraphs and good pairs in these when there are three 2-cycles.

Therefore we can assume that D has no directed 2-cycle and hence one can label the vertices of V_1 by a_1, b_1, c_1 and the vertices of V_2 by a_2, b_2, c_2 so that a_1(resp. b_1, c_1) is not adjacent to a_2 (resp. b_2, c_2) and adjacent to the two other vertices and for every $i, j, k \in [2]$, $D\langle\{a_i, b_j, c_k\}\rangle$ is a tournament.

In particular, every vertex of D has in- and out-degree exactly 2.

If D_1 and D_2 are both directed 3-cycles, then $D \setminus (A(D_1) \cup A(D_2))$ is a directed 6-cycle C. Let a be an arc of C from V_1 to V_2, let also P_1 be a hamiltonian directed path of D_1 ending in the tail of a and P_2 a hamiltonian directed path of D_2 starting at the head of a. Then $(C \setminus a, P_1 + a + P_2)$ is a good pair of D.

If one of the D_i, say D_1, is not a directed cycle, then D must be one of the three digraphs depicted in Figure 8, and so D has a good pair. □

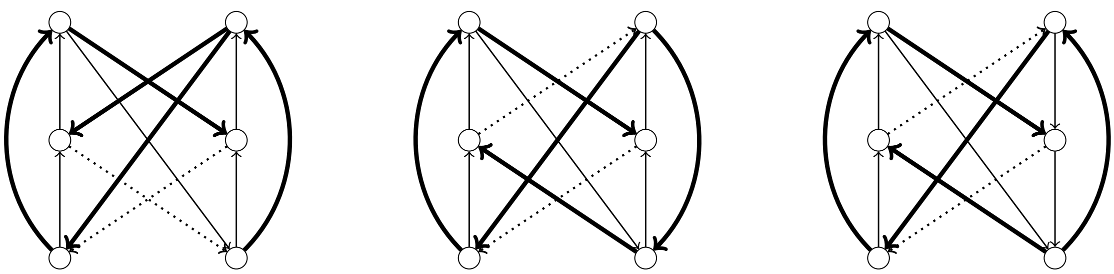

Figure 8: Good pairs of three co-bipartite digraphs of order 6.

6 Proof of Theorem 3

Now we are ready to prove our main result which we recall for easy reference.

Theorem 3 If D is a digraph with $\alpha(D) \leq 2 \leq \lambda(D)$, then D has a good pair.

Proof. For the sake of contradiction, assume that D has no good pair. By Proposition 19, $|V(D)| \geq 6$.

Claim 21.1. *There is no $Q \subseteq V(D)$ such that $D\langle Q\rangle$ has a good pair and every vertex in $V(D)\setminus Q$ is adjacent to at least one vertex in Q.*

Proof. Suppose to the contrary that there exists $Q \subseteq V(D)$ such that $D\langle Q\rangle$ has a good pair (I', O'), and every vertex in $V(D) \setminus Q$ is adjacent to at least one vertex in Q. Furthermore assume that $|Q|$ is maximum with this property. Let $X = N^+_D(Q)$ and let $Y = N^-_D(Q)$. By the maximality of Q and Lemma 16, $X \cap Y = \emptyset$.

Let X_i, $i \in [a]$, be the terminal strong components in $D\langle X\rangle$ and let Y_j, $j \in [b]$, be the initial strong components in $D\langle Y\rangle$. As $\alpha(D) \leq 2$ we note that $1 \leq a, b \leq 2$.

As $\lambda(D) \geq 2$ there are at least two arcs from X_i (for each $i \in [a]$) to Y and at least two arcs from X to Y_j (for each $j \in [b]$). Let x_1y be an arbitrary arc out of X_1 ($y \in Y$). If $y \in Y_1 \cup Y_b$, then without loss of generality assume that $y \in Y_1$. Let $P_1 = \{x_1y\}$. Therefore exists an arc, xy_1 from X to Y_1 which is different from x_1y (as Y_1 has at least two arcs into it) and we take $P_2 = \{xy_1\}$. If $a = 2$, then we let x_2y' be any arc out of X_2, which is different from xy_1 and we add x_2y' to P_1. If $b = 2$, then we let $x'y_2$ be any arc into Y_2 which is different from x_2y' (and which by the definition of x_1y is also different from x_1y) and we add $x'y_2$ to P_2.

Let D_X be the digraph obtained from $D\langle X\rangle$ by adding one new vertex y^* and arcs from x_i to y^* for $i \in [a]$. Note that $\mathrm{In}(D_X) = \{y^*\}$ and thus there exists an in-branching I_X in D_X with root y^*. Set $T_X = I_X \setminus y^*$. Analogously let D_Y be equal to $D\langle Y\rangle$ after adding one new vertex x^* and arcs from x^* to y_i for $i \in [b]$. Note that $\mathrm{Out}(D_Y) = \{x^*\}$ and thus there exists an out-branching O_Y in D_Y with root x^*. Set $T_Y = O_Y \setminus x^*$.

Now let I be the in-branching of D obtained from I' as follows. For each $u \in Y$ we add u and an arc from u to Q to I. We then add to I the digraph T_X and the arcs in P_1.

Let O be the out-branching of D obtained from O' as follows: For each $u \in X$ we add u and an arc from Q to u to O; we then add to O the digraph T_Y and the arcs in P_2. By construction, I and O are arc-disjoint so (I, O) is a good pair in D, a contradiction. $\Diamond$

Claim 21.2. *There is no $Q \subseteq V(D)$, such that $D\langle Q\rangle$ is not semicomplete but does have a good pair.*

Proof. Suppose to the contrary that there exists a $Q \subseteq V(D)$, such that $D\langle Q\rangle$ is not semicomplete but does have a good pair (I', O'). Let u and v be non-adjacent vertices in Q. Let $w \in V(D) \setminus Q$ be arbitrary. Note that w is adjacent to u or to v (or both) as $\alpha(D) \leq 2$. Therefore every vertex in $V(D) \setminus Q$ is adjacent to at least one vertex in Q. This contradicts Claim 21.1. $\Diamond$

Let R be a largest clique in D.

Claim 21.3. $|R| = 3$.

Proof. Using Ramsey theory, it is well-known that every digraph of order at least 6 either has an independent set of size 3 or a clique of order 3. As $\alpha(D) \leq 2$, we have $|R| \geq 3$. Suppose to the contrary that $|R| \geq 4$.

Let $X = N^+_D(R)$ and let $Y = N^-_D(R)$ and let $Z = V(D) \setminus (R \cup X \cup Y)$. For the sake of contradiction, assume that there exists $w \in X \cap Y$. By Corollary 15 and Lemma 16, there exists a good pair in $D\langle R \cup \{w\}\rangle$. Furthermore $D\langle R \cup \{w\}\rangle$ is not semicomplete, as R is a largest clique. This contradicts Claim 21.2. So $X \cap Y = \emptyset$.

We distinguish two cases depending on whether or not $D\langle R\rangle$ is strongly connected.

Case A. *$D\langle R\rangle$ is strongly connected.*

Assume that some vertex $x \in X$ has two arcs, say r_1x and r_2x, into it from R. We will first show that there is either a good r_1-pair or a good r_2-pair in $D\langle R\rangle$. Without loss of generality r_1 is an in-neighbour of r_2. Assume for a contradiction that there is no good r_1-pair and no good r_2-pair in $D\langle R\rangle$. As $D\langle R\rangle$ is strongly connected, we have $In(D\langle R\rangle) = R$ and, by Theorem 14, both (R, r_1) and (R, r_2) are exceptions.

Therefore $N^-_{D\langle R\rangle}(r_1) = \{r'\}$ for some $r' \in V(R)$ and $d^-_R(r') = 1$ and $N^-_R(r_2) = \{r_1\}$ and $d^-_{D\langle R\rangle}(r_1) = 1$. This implies that $|R| = 3$, contradiction. This contradiction implies that there is a good r_1-pair or a good r_2-pair in $D\langle R\rangle$. Without loss of generality assume that there is a good r_1-pair, (I_R, O_R), in $D\langle R\rangle$. Then $(I_R + r_1x, O_R + r_2x)$ is a x-good pair in $D\langle R \cup \{x\}\rangle$. Furthermore $D\langle R \cup \{x\}\rangle$ is not semicomplete, as R is a largest clique. This contradicts Claim 21.2. Therefore every $x \in X$ has exactly one arc into it from R. Analogously every $y \in Y$ has exactly one arc out of it to R.

Consequently, every vertex in $V(D)\backslash R$ is adjacent to at most one vertex in R. Therefore, if $u, v \in V(D)\backslash R$ are non-adjacent in D then there exists $r \in R$ such that $\{u, v, r\}$ is an independent set, a contradiction. Hence $V(D) \setminus R$ is semicomplete and D is co-bipartite. Thus, by Theorem 21, D has a good pair, a contradiction. This completes the proof of Case A.

Case B. *$D\langle R\rangle$ is not strongly connected.*

Let $R_1, R_2, \ldots, R_l$ denote the strong components of $D\langle R\rangle$, where $l \geq 2$, such that $R_i \mapsto R_j$ for all $1 \leq i < j \leq l$. As $\lambda(D) \geq 2$, there are at least two arcs out of R_l in D. Let r_lx and r'_lx' be two such arcs and note that $x, x' \in X$. Analogously let yr_1 and $y'r'_1$ denote two arcs from Y to R_1. By Lemma 11, R has a good pair so it follows from the the maximality of R and Claim 21.2 x, x' have no out-neighbour in $V(R)$ and y, y' have no in-neighbour in $V(R)$.

Assume that there exists an arc rx from R to x which is distinct from r_lx. By Lemma 11, there exists a good r_l-pair (I_R, O_R) of $D\langle R\rangle$, and so $(I_R + r_lx, O_R + rx)$ is a good pair in $D\langle R \cup \{x\}\rangle$. This contradicts Claim 21.2.

Therefore x is adjacent to exactly one vertex in R (namely r_l). Analogously x', y and y' are each adjacent with exactly one vertex in R. As $\alpha(D) \leq 2$ and $|V(R)| \geq 4$, this implies that $D' = D\langle\{x, x', y, y'\}\rangle$ is a digraph of order 4 which must be semicomplete. So by Corollary 15, there exists a good pair (I', O') in D'. Moreover by Lemma 11, $D\langle R\rangle$ has a good (r_l, r_1)-pair (I_R, O_R). Now $(I' \cup I_R + r_lx, O' \cup O_R + yr_1)$ is a good pair for $D\langle R \cup V(D')\rangle$. This contradicts Claim 21.2.

$\Diamond$

Claim 21.4. *No subdigraph of D of order at least 4 has a good pair.*

Proof. Suppose to the contrary that a subdigraph D' of D of order at least 4 has a good pair. If D' is semicomplete, then it contradicts Claim 21.3, and if D' is not semicomplete, then it contradicts Claim 21.2. $\Diamond$

Claim 21.5. *All directed 2-cycles are vertex disjoint.*

Proof. Suppose to the contrary that two directed 2-cycles intersect, say (x, y, x) and (x, z, x) are both directed 2-cycles in D. Let $X = \{x, y, z\}$. Let $I = (y, x, z)$ and $O = (z, x, y)$. Note that (I, O) is a good pair for $D\langle X\rangle$.

Let $w \in V(D) \setminus X$ be arbitrary. The vertex w is not adjacent to both x and y for otherwise by Proposition 17, $D\langle\{x, y, w\}\rangle$ has a good pair and so by Lemma 16, $D\langle\{x, y, z, w\}\rangle$ has a good pair, a contradiction to Claim 21.4. Similarly, w is not adjacent to both x and z.

Assume now for a contradiction that w is adjacent to y and z. We will again show that $D\langle\{x, y, z, w\}\rangle$ has a good pair, a contradiction to Claim 21.4. If $wy, zw \in A(D)$ or $yw, wz \in A(D)$, then $D\langle X \cup \{w\}\rangle$ has a good pair by Lemma 16, because $D\langle X\rangle$ has a good pair. If $wy, wz \in A(D)$, then $((w, y, x, z); (w, z, x, y))\}$ is a good pair of $D\langle X \cup \{w\}\rangle$. If $yw, zw \in A(D)$, then $((y, x, z, w); (z, x, y, w))$ is a good pair of $D\langle X \cup \{w\}\rangle$. Therefore w is adjacent to at most one vertex in X.

If $w, w' \in V(D) \setminus X$, then w and w' must be adjacent, since otherwise there is a vertex in X which together with $\{w, w'\}$ forms an independent set of size 3, a contradiction. Therefore $D \setminus X$ is a semicomplete digraph.

Now $D\langle X\rangle$ is not semicomplete for otherwise D is co-bipartite, a contradiction to Theorem 21. But (I, O) is a good pair of $D\langle X\rangle$, a contradiction to Claim 21.2. $\Diamond$

Claim 21.6. *$D\langle R\rangle$ is a tournament.*

Proof. Suppose for a contradiction that $D\langle R\rangle$ contains a 2-cycle (x, y, x). Let $R = \{x, y, z\}$.

We distinguish several cases depending on the arcs between z and $\{x, y\}$.

Case A. $zx, zy \in A(D)$.

As $\lambda(D) \geq 2$ there are at least two arcs leaving the set $\{x, y\}$ in D. Let u_1v_1 and u_2v_2 be two such arcs. By Claim 21.5 we have $z \notin \{v_1, v_2\}$.

If $v_1 = v_2$, then, without loss of generality, we may assume that $u_1v_1 = xv$ and $u_2v_2 = yv$ (where $v = v_1 = v_2$). Now let $I' = (z, x, y, v)$ and $O' = (z, y, x, v)$ and note that (I', O') is a good pair for $D\langle R \cup \{v\}\rangle$. But $D\langle R \cup \{v\}\rangle$ is not semicomplete by our choice of R. This contradicts Claim 21.2. So $v_1 \neq v_2$.

As $\lambda(D) \geq 2$ there are at least two arcs entering z in D. Let r_1z and r_2z be two such arcs. Note that $r_1 \neq r_2$ and that by Claim 21.5 we have $r_1, r_2 \notin \{x, y\}$.

Let $I = (z, x, y)$ and $O = (z, y, x)$ and note that (I, O) is a good pair in R. By Lemma 16 and Claim 21.4 we note that there are no arcs from R to $\{r_1, r_2\}$ and there are no arcs from $\{v_1, v_2\}$ to R. Therefore $Y = \{r_1, r_2, v_1, v_2\}$ is a set of four distinct vertices.

We will now show that Y is a clique, a contradiction to the maximality of R. We will do this by showing that every vertex in Y has at most one neighbour in R. This will imply the claim as $\alpha(D) = 2$.

- If a vertex in $\{r_1, r_2\}$ is adjacent to a vertex in $\{x, y\}$ then assume without loss of generality that r_1 is adjacent to x, which implies that $r_1x \in A(D)$, by the above observation. Let $I' = \{r_1x, xy, zx\}$ and $O' = \{r_1z, zy, yx\}$ and note that (I', O') is a good pair for $D\langle X \cup \{r_1\}\rangle$, contradicting Claim 21.4. Therefore no vertex from $\{r_1, r_2\}$ is adjacent to a vertex in $\{x, y\}$ and so the vertices r_1 and r_2 are adjacent to exactly one vertex in R.

- Now assume for the sake of contradiction that a vertex of $\{v_1, v_2\}$, say v_1, is adjacent to at least two vertices in R. The vertex v_1 is not adjacent to x and y, for otherwise, we could have let $v_1 = v_2$, contradicting the arguments above. So we may assume that v_1 is adjacent to x and z, so $xv_1, zv_1 \in A(D)$. Let $I' = (z, y, x, v_1)$ and $O' = (z, x, y) + zv_1$ and note that (I', O') is a good pair for $D\langle X \cup \{v_1\}\rangle$, contradicting Claim 21.4. Therefore every vertex in $\{v_1, v_2\}$ is adjacent to at most one vertex in R.

This completes the Case A.

The case when $xz, yz \in A(D)$ is proved analogously to Case A by reversing all arcs.

Case B. $zx, yz \in A(D)$.

Let $I = (z, x) + yx$ and $O = (x, y, z)$ and note that (I, O) is a good pair for R. Let $w \in V(D) \setminus R$ be arbitrary. Note that w cannot have an arc to R and an arc from R by Lemma 16 and Claim 21.4. For the sake of contradiction assume that w is adjacent to at least two of the vertices in R. If w is adjacent to x and y, then we would be in the above case (as $wx, wy \in A(D)$ or $xw, yw \in A(D)$). We may, without loss of generality, assume that $wz, wx \in A(D)$. Let $I' = (w, z, x) + yx$ and $O' = (w, x, y, z)$ and note that (I', O') is a good pair for $D\langle R \cup \{w\}\rangle$ contradicting Claim 21.4. Therefore w has at most one neighbour in R and so $V(D) \setminus R$ is a clique. Hence D is co-bipartite, a contradiction to Theorem 21. This completes the Case B.

The case when $xz, zy \in A(D)$ is proved analogously to Case B by reversing all arcs. This completes the proof of Claim 21.6. $\Diamond$

Claim 21.7. *Either D has no directed 2-cycles or D has exactly one directed 2-cycle and $|V(D)| = 7$.*

Proof. Assume for a contradiction that D has a directed 2-cycle (x, y, x). Let X (resp. Y) be the set of vertices in $V(D) \setminus \{x, y\}$ adjacent to x (resp. y). Since $\delta^0(D) \geq 2$, x (resp. y) has an in-neighbour and an out-neighbour in X (resp. Y) and they cannot be the same by Claim 21.5. So $|X|, |Y| \geq 2$. By Claim 21.6,

$X \cap Y = \emptyset$. As no vertex in X is adjacent to y (resp. x), X (resp. Y) is a clique and so $X \cup \{x\}$ (resp. $Y \cup \{y\}$) is a clique, implying that $|X| = |Y| = 2$ by Claim 21.3.

Let $Z = V(D) \setminus (\{x, y\} \cup X \cup Y)$. Then Z is non-empty for otherwise D would be co-bipartite and hence have a good pair by Theorem 21. Now $X \cup Z$ and $Y \cup Z$ are cliques since $\alpha(D) \leq 2$, so $|Z| = 1$ by Claim 21.3. Hence D has 7 vertices. Moreover we check that every pair of vertices of D except $\{x, y\}$ is in the neighbourhood of a third vertex, and then cannot induce a directed 2-cycle in D by Claim 21.6. $\Diamond$

Now we are ready to finish the proof of Theorem 3. Note that $n \geq 7$ since every digraph D on at most 6 vertices with $\alpha(D) = 2$, $\delta^0(D) \geq 2$ and no 2-cycle is co-bipartite. It also follows from Theorem 9 and Claim 21.3 that $n \leq 8$.

By Theorem 8, D has a directed hamiltonian path P. Let x and y be its initial and terminal vertex, respectively. Let D' be the digraph that we obtain by deleting all the arcs of P. If D' has precisely one initial strong component, then it has an out-branching B^+ so (P, B^+) is a good pair. Similarly, if D' has only one terminal strong component, then it has an in-branching B^- and (B^-, P) is a good pair. Hence we may assume that D' is not strongly connected and that it has at least two initial components D'_1, D'_2 and at least two terminal components D'_3, D'_4. As $\delta^0(D) \geq 2$ we have $\delta^0(D') \geq 1$, implying that $|D'_i| \geq 3$ for $i \in [4]$ if D has no directed 2-cycle. Now Claim 21.7 and the fact that $n \leq 8$ implies that D' has exactly two strong components D'_1, D'_2 and there are no arcs between these in D'. This means that every arc of D that goes between $V(D'_1)$ and $V(D'_2)$ belongs to P.

Since $n \geq 7$ we may assume w.l.o.g. that $|D'_1| \geq 4$. By Claim 21.4 and Corollary 15, $D\langle V(D'_1)\rangle$ is not semicomplete. The digraph D'_2 has also order at least 2 and if it has order at least 4 then $D\langle V(D'_2)\rangle$ is not semicomplete.

Suppose first that $|D'_1| = |D'_2| = 4$ (in which case D has no directed 2-cycle by Claim 21.7). W.l.o.g. $x \in V(D'_1)$ so x has in-degree at least 2 in D'_1, implying that D'_1 has precisely 5 arcs (it cannot have more since then it would be semicomplete, contradicting Claim 21.3) and hence P uses no arc in $D\langle V(D'_1)\rangle$ (if it did use an arc inside $V(D'_1)$, then that arc would be in a 2-cycle in D, contradicting Claim 21.7 since $n = 8$ when $|D'_1| = |D'_2| = 4$). Let x^+ be the successor of x on P and note that $x^+ \in V(D'_2)$.

Let us first observe that D'_1 has an out-branching B_1^+ that does not use all arcs out of x. This is clear if D'_1 is hamiltonian so assume it is not. Then D'_1 is the digraph with vertex set z_1, z_2, z_3, z_4 and arcs $z_1z_3, z_1z_4, z_2z_1, z_3z_2, z_4z_2$. Now let $B^+_{x^+}$ be an out-branching of D'_2 rooted at x^+ and let xz be an arc out of x in D'_1 which is not in B_1^+. Then we obtain a good pair (I, O) by letting $I = P \setminus xx^+ + xz$ and $O = B_1^+ \cup B^+_{x^+} + xx^+$, a contradiction.

Assume now that $|D'_2| = 3$. Then $V(D'_2)$ is a clique, and so $D\langle V(D'_2)\rangle$ has no directed 2-cycles by Claim 21.6. Thus $D'_2 = D\langle V(D'_2)\rangle$, D'_2 is a directed 3-cycle and P does not use any arc inside $D\langle V(D'_2)\rangle$. Label the vertices of $V(D'_2)$ by a, b, c so that P visits these vertices in that order. Let a^+ (resp. b^+) be the successor of a (resp. b) on P. If D'_2 is the directed 3-cycle (a, c, b, a), then we obtain a good pair (I, O) (and a contradiction) by letting $I = P \setminus aa^+ + ac$ (so the root will be y) and $O = (c, b, a, a^+) \cup B^+_{a^+}$, where $B^+_{a^+}$ is any out-branching rooted at a^+ in D'_1. If D_2 is the 3-cycle (a, b, c, a) then we obtain a good pair (I, O) (and a contradiction) by letting $I = P \setminus bb^+ + bc$ and $O = (c, a, b, b^+) \cup B^+_{b^+}$, where $B^+_{b^+}$ is any out-branching rooted at b^+ in D'_1.

Assume finally that $|D'_2| = 2$. Then D'_2 is a directed 2-cycle (a, b, a). Without loss of generality, we may assume that P visits a before b. Let a^+ be successor of a on P. Letting $I = P \setminus aa^+ + ab$ and $O = (b, a, a^+) \cup B^+_{a^+}$, where $B^+_{a^+}$ is any out-branching rooted at a^+ in D'_1. Then (I, O) is a good pair of D, a contradiction. This completes the proof of Theorem 3. $\square$

7 Digraphs with bounded independence number and no good pair

The following example shows that $\alpha(D) = 2 = \lambda(D)$ is not sufficient to guarantee a pair of arc-disjoint branchings B_s^+, B_t^- for every choice of vertices $s, t \in V(D)$. Let H_1 be the strong semicomplete digraph on four vertices a_1, b_1, c_1, d_1 that we obtain from the directed 4-cycle $(a_1, b_1, c_1, d_1, a_1)$ by adding the arcs of the directed 2-cycle (a_1, c_1, a_1) and the arc d_1b_1. Let H_2 be the strong semicomplete digraph on four vertices a_2, b_2, c_2, d_2 that we obtain from the directed 4-cycle $(a_2, d_2, c_2, b_2, a_2)$ by adding the arcs of the directed 2-cycle (a_2, c_2, a_2) and the arc b_2d_2. The digraph W is obtained from the disjoint union of H_1 and H_2 by adding the arcs of the directed 4-cycle $(d_1, d_2, b_1, b_2, d_1)$. See Figure 9. It is easy to verify that D is 2-arc-strong.

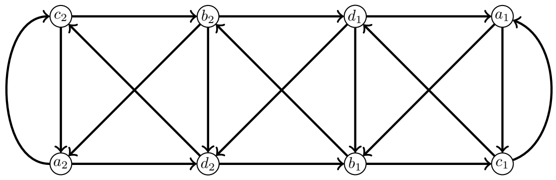

Figure 9: The 2-arc-strong digraph W

Proposition 22. *The digraph W has no pair of arc-disjoint branchings $B^+_{c_2}, B^-_{c_1}$.*

Proof. Suppose that such branchings do exist. We first consider the case when the arc c_2b_2 is in $B^+_{c_2}$. Then the arc c_2a_2 is in $B^-_{c_1}$ and the arc b_2a_2 is in $B^+_{c_2}$. The set $\{c_2, a_2\}$ shows that the arc a_2d_2 is in $B^-_{c_1}$ and the set $\{c_2, a_2, d_2\}$ shows that d_2b_1 is in $B^-_{c_1}$. Now the set $\{c_2, a_2, b_2, d_2\}$ shows that the arc b_2d_1 is in $B^+_{c_2}$. This implies that the arc b_2d_2 is in $B^-_{c_1}$. Next the set $\{a_2, b_2, c_2, d_2, b_1\}$ shows that the arc b_1c_1 must belong to $B^-_{c_1}$ and the set $\{a_2, b_2, c_2, d_2, d_1, b_1\}$ shows that the arc d_1a_1 must be in $B^+_{c_2}$. Then arc a_1c_1 must belong to $B^+_{c_2}$, the arc a_1b_1 must belong to $B^-_{c_1}$ and the arc d_1b_1 must belong to $B^+_{c_2}$. Now all out-going arcs of d_1 were added to $B^+_{c_2}$, contradiction.

Suppose next that the arc c_2a_2 belongs to $B^+_{c_2}$. Then analogously to the argument above we conclude that the arcs c_2b_2, b_2d_1 all belong to $B^-_{c_1}$ and the arcs a_2d_2, d_2b_1, b_1c_1 all belong to $B^+_{c_2}$. This implies that the arc b_1b_2 belongs to $B^+_{c_2}$ but then both arcs leaving the vertex b_1 are in in $B^+_{c_2}$, contradiction. □

Proposition 23. *There are infinitely many digraphs with arc-connectivity 2 and independence number 3 which do not have arc-disjoint branchings B^+_s, B^-_t for some choice of vertices $s, t \in V$.*

Proof. For $n \geq 9$ let $\mathcal{W}'_n$ be the class of digraphs that we obtain from a strong semicomplete digraph S on $n - 8$ vertices and a copy of the digraph W above by adding all possible arcs from $V(S)$ to c_2 and all possible arcs from c_1 to $V(S)$. It is easy to check that every digraph in $\mathcal{W}'_n$ is 2-arc-strong and has independence number 3. We claim that no digraph in $\mathcal{W}'_n$ has pair of arc-disjoint branchings B^+_s, B^-_t where $s, t \in V(S)$. Suppose that such a digraph W'_n had arc-disjoint branchings B^+_s, B^-_t. Then the restriction of these branchings to $V(W)$ would be an out-branching rooted at c_2 and an in-branching rooted at c_1 which are arc-disjoint, contradicting Proposition 22. □

The following result shows that there is no function $f(k)$ with the property that every 2-arc-strong digraph $D = (V, A)$ with $\alpha(D) = k$ and $|V| \geq f(k)$ has a good pair.

Theorem 24. *There exist infinitely many digraphs with arc-connectivity 2 and independence number at most 7 which have no good pair.*

Proof. Let S be an arbitrary strong semicomplete digraph on $n \geq 1$ vertices and let W_S be the digraph on $n + 24$ vertices that we obtain from S and three copies of the digraph W from Proposition 22 by adding all possible arcs from $V(S)$ to the three copies of the vertex c_2 and all possible arcs from the three copies of the vertex c_1 to $V(S)$. Then W_s is 2-arc-strong and has independence number 7. We claim that W_s has no out-branching which is arc-disjoint from some in-branching. Suppose such a pair B^+_s, B^-_t did exist. Then at least one copy of W would contain none of s, t and hence the restriction of B^+_s, B^-_t to that copy would be pair of arc-disjoint branchings $B^+_{c_2}, B^-_{c_1}$, contradicting that W has no such pair. □

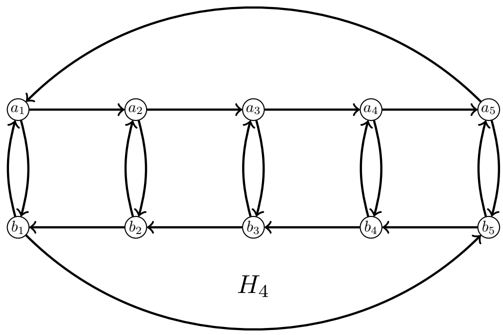

Figure 10: The digraph H_4 with $\alpha(H_4) = 4$, $\lambda(H_4) = 2$ and no good pair.

Proposition 25. *The digraph H_4 in Figure 10 has $\alpha(H_4) = 4$, $\lambda(H_4) = 2$ and no good pair.*

Proof. First observe that for each $i \in [5]$ the subdigraph $H_4\langle\{a_i, a_{i+1}, b_i, b_{i+1}\}\rangle$, where $a_6 = a_1, b_6 = b_1$ induces a copy of the digraph E_4. By Proposition 18, E_4 has no good pair. Now suppose that H_4 has a good pair (I, O). Then these branchings must avoid at least one arc inside each of the five copies of E_4 (otherwise the restriction of (I, O) to such a copy would be a good pair in E_4). But H_4 has 20 arcs, 18 of which must belong to either I or O and there is pair of arcs with at least one in each of the five copies of E_4, contradiction. □

Proposition 26. *Every digraph on 6 vertices and arc-connectivity at least 2 has a good pair.*

Proof. Let D have 6 vertices and $\lambda(D) \geq 2$ and suppose that D has no good pair. By Theorem 3 we may assume that $\alpha(D) \geq 3$. If $\alpha(D) = 4$ then D contains, as a spanning subdigraph, the digraph that we obtain from the complete bipartite graph $K_{2,4}$ by replacing each edge by a directed 2-cycle and it is easy to check that this has a good pair. So we can assume that $\alpha(D) = 3$. Let $X = \{x_1, x_2, x_3\}$ be an independent set of size 3. Then each x_i is incident to a directed 2-cycle (x_i, y_i, x_i). By Lemma 20, we can assume that D does not contain a subdigraph on 3 vertices with a good pair, so by Proposition 17 we conclude that $|\{y_1, y_2, y_3\}| = 3$ and that $d^+(x_i) = d^-(x_i) = 2$ for $i \in [3]$. Let A' contain all arcs between X and $V(D) \setminus X$ except the arcs x_iy_i and y_ix_i for $i = 1, 2, 3$. Note that $|A'| = 6$. If the arcs of A' form a directed 6-cycle, then without loss of generality this is the 6-cycle $x_1y_2x_3y_1x_2y_3x_1$ and now D contains the spanning subdigraph D' in Figure 11 where we only show the arcs of a good pair. So we may assume that $V(D) \setminus X$ has a vertex with in-degree 2 and another with out-degree 2 wrt the arcs A'. W.l.o.g. y_1 has in-degree 2 and y_3 has out-degree 2 wrt A'. Now D contains the spanning subdigraph D'' shown in the right part of Figure 11 together with a good pair.

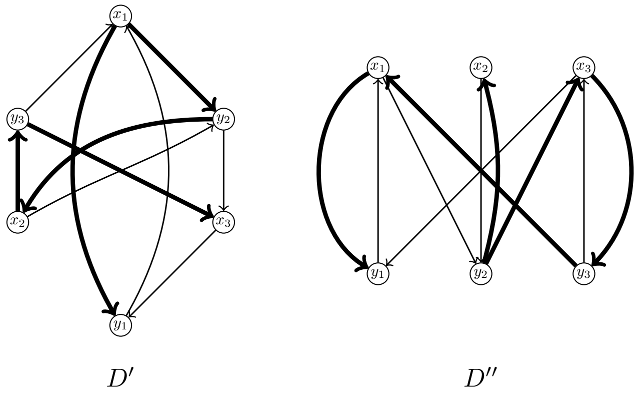

Figure 11: The two possible digraphs D' and D'' with good pairs.

□

8 Remarks and open problems

Problem 27. What is the smallest number n of vertices in a 2-arc-strong digraph which has no good pair?

By Proposition 26 and the example H_4 in Figure 10 we know that $7 \leq n \leq 10$.

The infinite family $\mathcal{W}'_n$ in the proof of Proposition 23 shows that there are infinitely many 2-arc-strong digraphs with independence number 3 which have only a linear number of pairs s, t for which arc-disjoint branchings B_s^+, B_t^- exists (for each W'_n with $n \geq 10$ we can take $t \in V(S)$ arbitrary and let $s = b_1$). This leads to the following question.

Problem 28. Does there exist a digraph with independence number 3 and arc-connectivity 2 without a good pair?

Conjecture 29. Every 2-arc-strong digraph $D = (V, A)$ with $\alpha(D) = 2$ has a pair of arc-disjoint branchings B_s^+, B_s^- for every choice of $s \in V$.

Figure 12 shows that Conjecture 29 does not hold for directed multigraphs (the example is Figure 4 in [2]).

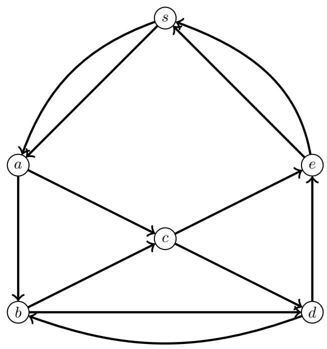

Figure 12: A 2-arc-strong multigraph D with $\alpha(D) = 2$ and no pair of arc-disjoint branchings B_s^+, B_s^-.

Conjecture 30. Every 3-arc-strong digraph $D = (V, A)$ with $\alpha(D) = 2$ has a pair of arc-disjoint branchings B_s^+, B_t^- for every choice of $s, t \in V$.

Problem 31. What is the complexity of deciding whether a digraph $D = (V, A)$ with $\alpha(D) = 2$ has an out-branching and an in-branching that are arc-disjoint?

Problem 32. What is the complexity of deciding whether a digraph $D = (V, A)$ with $\alpha(D) = 2$ has an out-branching B_s^+ and in-branching B_t^- that are arc-disjoint when $s, t \in V$ are prescribed?

It was shown in [9] that one can decide in polynomial time whether a digraph of independence number 2 has arc-disjoint paths P_1, P_2, where P_i is an (s_i, t_i)-path for $i = 1, 2$, where s_1, s_2, t_1, t_2 are part of the input. This suggests that Problems 31 and 32 could be polynomial-time solvable.

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References

[1] J. Bang-Jensen. Edge-disjoint in- and out-branchings in tournaments and related path problems. *J. Combin. Theory Ser. B*, 51(1):1–23, 1991.

[2] J. Bang-Jensen. Problems and conjectures concerning connectivity, paths, trees and cycles in tournament-like digraphs. *Discrete Math.*, 309:5655–5667, 2009.

[3] J. Bang-Jensen and Gutin. G (eds.). *Classes of Directed Graphs*. Springer Monographs in Mathematics. Springer Verlag, London, 2018.

[4] J. Bang-Jensen and G. Gutin. *Digraphs: Theory, Algorithms and Applications*. Springer-Verlag, London, 2nd edition, 2009.

[5] J. Bang-Jensen and J. Huang. Decomposing locally semicomplete digraphs into strong spanning subdigraphs. *J. Combin. Theory Ser. B*, 102:701–714, 2010.

[6] J. Bang-Jensen and S. Simonsen. Arc-disjoint paths and trees in 2-regular digraphs. *Discrete Applied Mathematics*, 161(16-17):2724–2730, 2013.

[7] C.C. Chen and P. Manalastas, Jr. Every finite strongly connected digraph of stability 2 has a Hamiltonian path. *Discrete Math.*, 44(3):243–250, 1983.

[8] J. Edmonds. Edge-disjoint branchings. In *Combinatorial Algorithms*, pages 91–96. Academic Press, 1973.

[9] A. Fradkin and P.D. Seymour. Edge-disjoint paths in digraphs with bounded independence number. *J. Combin. Theory Ser. B*, 110:19–46, 2015.

[10] L. Lovász. On two min–max theorems in graph theory. *J. Combin. Theory Ser. B*, 21:96–103, 1976.

[11] J.W. Moon. On subtournaments of a tournament. *Can. Math. Bull.*, 9:297–301, 1966.

[12] C.St.J.A. Nash-Williams. Edge-disjoint spanning trees of finite graphs. *J. London Math. Soc.*, 36:445–450, 1961.

[13] C. Thomassen. Configurations in graphs of large minimum degree, connectivity, or chromatic number. *Annals of the New York Academy of Sciences*, 555:402–412, 1989.

[14] W.T. Tutte. On the problem of decomposing a graph into n connected factors. *J. London Math. Soc.*, 36:221–230, 1961.