Further extensions of the Grötzsch Theorem

Xuan Hoang La, Borut Lužar, Kenny Štorgel

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Abstract

The Grötzsch Theorem states that every triangle-free planar graph admits a proper 3-coloring. Among many of its generalizations, the one of Grünbaum and Aksenov, giving 3-colorability of planar graphs with at most three triangles, is perhaps the most known. A lot of attention was also given to extending 3-colorings of subgraphs to the whole graph. In this paper, we consider 3-colorings of planar graphs with at most one triangle. Particularly, we show that precoloring of any two non-adjacent vertices and precoloring of a face of length at most 4 can be extended to a 3-coloring of the graph. Additionally, we show that for every vertex of degree at most 3, a precoloring of its neighborhood with the same color extends to a 3-coloring of the graph. The latter result implies an affirmative answer to a conjecture on adynamic coloring. All the presented results are tight.

Keywords: Grötzsch Theorem, planar graph, 3-coloring, precoloring extension, one triangle

1 Introduction

A *proper coloring* of a graph G is an assignment of colors to its vertices such that adjacent vertices are assigned distinct colors. For an integer k, a graph is k-colorable if it admits a proper coloring with at most k colors; the smallest such k is called the *chromatic number* of G, denoted by $\chi(G)$.

The Four Color Theorem [5, 6] states that the chromatic number of any planar graph is at most 4, but determining which graphs achieve the equality is an NP-complete problem [14]. Consequently, searching for properties of (planar) graphs that guarantee 3-colorability is a very vibrant field (see, e.g., [7] for a survey). It turns out that triangles play an important role in this decision problem; indeed, a cornerstone theorem of Grötzsch [21] states that every triangle-free planar graph is 3-colorable. Consequently, the focus in the field turned to investigating ways in which triangles can appear in 3-colorable planar graphs. For example, for any plane triangulation, Heawood [24] established a necessary and sufficient condition by showing that it is 3-colorable if and only if all of its vertices have even degrees (see [15, 19, 26] for generalizations of this statement).

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We may also allow triangles in general planar graphs and still retain 3-colorability: Havel [23] conjectured that a 3-colorable planar graph may contain arbitrarily many triangles as long as they are sufficiently far apart and Steinberg [29] conjectured that every planar graph without cycles of length 4 and 5 is 3-colorable. While Havel's conjecture has been proved by Dvořák, Kráľ, and Thomas [18], Steinberg's conjecture has been refuted by Cohen-Addad et al. [13]. Currently the best result of a similar flavor is due to Borodin et al. [9], stating that every planar graph without cycles of length 5 and 7, and without adjacent triangles is 3-colorable. On the other hand, there are 3-colorable planar graphs that may have close triangles (even incident) and have no short cycles forbidden: as proved in [16], every planar graph obtained as a subgraph of the medial graph of a bipartite plane graph is 3-colorable (in fact, 3-choosable).

Another direction of research is focused on planar graphs with small number of triangles. Grünbaum [22] noticed that a planar graph may contain three triangles and still retain 3-colorability. His original proof was incorrect and a corrected version was published by Aksenov [1].

Theorem 1.1 ([1])**.** *Every planar graph with at most three triangles is* 3*-colorable.*

Shorter proofs of this result were given by Borodin [11] and Borodin et al. [10]. The authors of the latter used the following result on 4-critical graphs due to Kostochka and Yancey [27].

Theorem 1.2 ([27])**.** *If* G *is a* 4*-critical graph on* n *vertices, then*

$$|E(G)| \geq \frac{5n - 2}{3}\,.$$

Theorem 1.2 is a restricted version of a more general theorem from [28], which describes k-critical graphs and was used in [8] to characterize all planar 4-critical graphs with exactly four triangles; we present these two results in Section 2.

Along with a short proof of Theorem 1.1, using Theorem 1.2, the authors of [10] presented short proofs of several other extensions of the Grötzsch Theorem, which guarantee 3-colorability of graphs being close to triangle-free planar graphs. In particular, they extended a result from [25] stating that a triangle-free planar graph with an additional vertex of degree 3 is also 3-colorable.

Theorem 1.3 ([10, 25])**.** *Let* G *be a triangle-free planar graph and let* H *be a graph such that* $G = H - v$ *for some vertex* v *of degree* 4 *of* H*. Then* H *is* 3*-colorable.*

They also gave a short proof of a precoloring extension result of Aksenov, Borodin, and Glebov [3].

Theorem 1.4 ([3, 10])**.** *Let* G *be a triangle-free planar graph. Then each coloring of any two non-adjacent vertices can be extended to a* 3*-coloring of* G*.*

Note that Theorem 1.4 extends the result of Aksenov [2] and Jensen and Thomassen [25] that a graph obtained from a triangle-free planar graph by adding one edge is 3-colorable.

From Theorems 1.3 and 1.4 one can derive another precoloring extension result.

Theorem 1.5 ([10]). *Let G be a triangle-free planar graph and let f be a face of G of length at most 5. Then each 3-coloring of f can be extended to a 3-coloring of G.*

On the other hand, if the face f has length k with $k \geq 6$, then not every precoloring of its vertices can be extended to a 3-coloring of G. The cases when $k = 6, 7, 8, 9$ were completely characterized in [20], [4], [17], and [12], respectively. Moreover, precoloring faces in planar graph of girth at least 5 have also been studied (see, e.g., [12] for more details).

In this paper, we introduce new results about 3-colorability of planar graphs with small number of triangles and some precolored vertices. First, we extend Theorem 1.4 to planar graphs with at most one triangle.

Theorem 1.6. *Let G be a planar graph with at most one triangle. Then each coloring of any two non-adjacent vertices can be extended to a 3-coloring of G.*

The result is tight in terms of the number of precolored vertices and in terms of the number of triangles; for example, the precolorings of graphs depicted in Figure 1 cannot be extended to a 3-coloring of the whole graph.
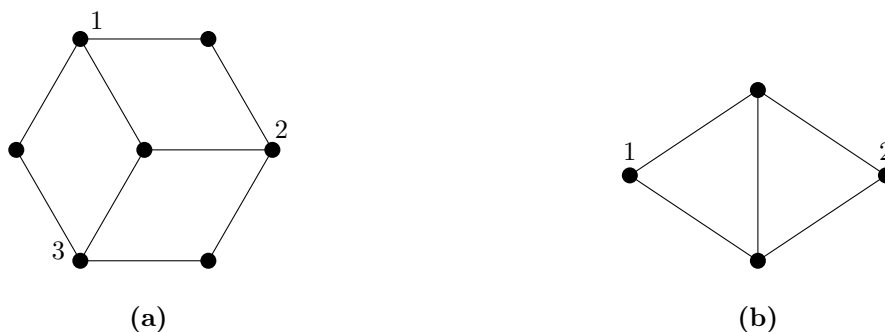

(a)   (b)

Figure 1: Not every precoloring of three vertices can be extended to a 3-coloring of a planar graph with at most one triangle (example (a)), nor can be every precoloring of two vertices in a planar graph with two triangles (example (b)).

As a corollary of Theorem 1.6, we obtain a theorem similar to Theorem 1.3 for planar graphs with at most one triangle.

Theorem 1.7. *Let G be a planar graph with at most one triangle and let H be a graph such that $G = H - v$ for some vertex v of degree at most 3 in H, which is adjacent with at most two vertices of the triangle in G if it exists. Then H is 3-colorable.*

Proof. Let $N(v) = \{v_1, v_2, v_3\}$. Without loss of generality, we may assume that v_1 and v_2 are not adjacent. Then, by Theorem 1.6, we can color v_1 and v_2 with the same color and so the three vertices in $N(v)$ will be colored with at most two colors, which means there is an available color for coloring v. $\square$

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Again, the result is tight in terms of the number of precolored vertices and in terms of the number of triangles (see Figure 2 for examples), as well as in terms of the number of neighbors of v on the triangle. Clearly, connecting v with all three vertices of the triangle would result in a subgraph isomorphic to K_4.
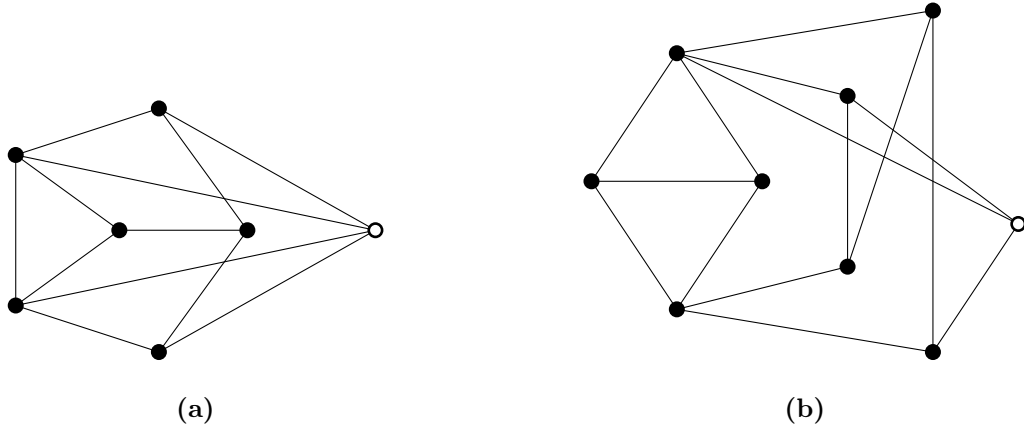

(a)   (b)

Figure 2: Not every graph obtained from a planar graph with at most one triangle by adding a 4-vertex is 3-colorable (example (a)), nor is a graph obtained from a planar graph with two triangles by adding a 3-vertex (example (b)). The added vertex is depicted with an empty disk in both cases.

Extending precolorings of small faces in planar graphs with one triangle is more restricted. We prove an analogue of Theorem 1.5 for faces of length at most 4.

Theorem 1.8. *Let G be a planar graph with at most one triangle and let f be a face of G of length at most 4. Then each 3-coloring of f can be extended to a 3-coloring of G.*

On the other hand, a precoloring of a 5-face in a planar graph with one triangle cannot always be extended to a 3-coloring of the whole graph; see example in Figure 3.
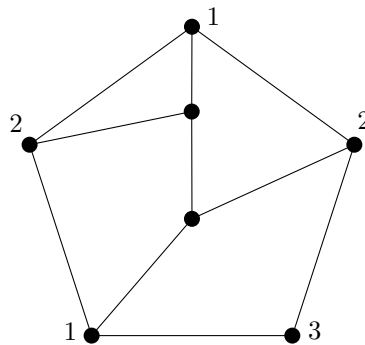

Figure 3: Precoloring of the outer 5-face which cannot be extended to a 3-coloring of the graph.

The result about extending a precoloring of an 8-cycle from [17] (as remarked in [12]) implies the following.

Theorem 1.9 ([17]). *Let G be a triangle-free planar graph and let v be a vertex of degree at most 4 in G. Then there exists a 3-coloring of G where all neighbors of v are colored with the same color.*

A similar result to Theorem 1.9 about coloring three neighbors of a vertex of an arbitrary degree can be obtained as a corollary of Theorem 1.3.

Corollary 1.10. *Let G be a triangle-free planar graph and let v_1, v_2, and v_3 be distinct vertices with a common neighbor v. Then there exists a 3-coloring of G where v_1, v_2, and v_3 are colored with the same color.*

Proof. By Theorem 1.3, the graph obtained from G by adding a 4-vertex x adjacent to v, v_1, v_2, and v_3 is 3-colorable. In its coloring, the vertices v_1, v_2, and v_3 are colored with the same color, since they must all be colored differently from v and x, which receive two distinct colors. □

We prove a somewhat weaker result for the case of planar graphs with one triangle. Let K_4' be the graph obtained from K_4 by subdividing once the three edges incident with a vertex v (see Figure 4). We call a graph K_4'-*free* if it does not contain K_4' as a subgraph in such a way that the vertex v of K_4' has degree 3 also in G. It is easy to see that the vertices in the neighborhood of v cannot be colored with a same color.
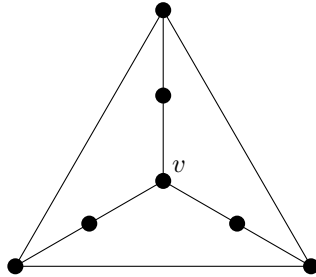
Figure 4: A planar graph with at most one triangle with a vertex v of degree 3 having an independent neighborhood $N(v)$ for which there is no 3-coloring such that all vertices in $N(v)$ receive the same color.

Theorem 1.11. *Let G be a K_4'-free planar graph with at most one triangle. Then, for every vertex of degree at most 3 with an independent neighborhood, a precoloring of its neighbors with the same color can be extended to a 3-coloring of G.*

Theorem 1.11 is tight in terms of the degree of a vertex and in terms of the number of triangles (see examples in Figure 5).

2 Preliminaries

In this section we present the terminology and the auxiliary results that we are using in the proofs of our theorems.

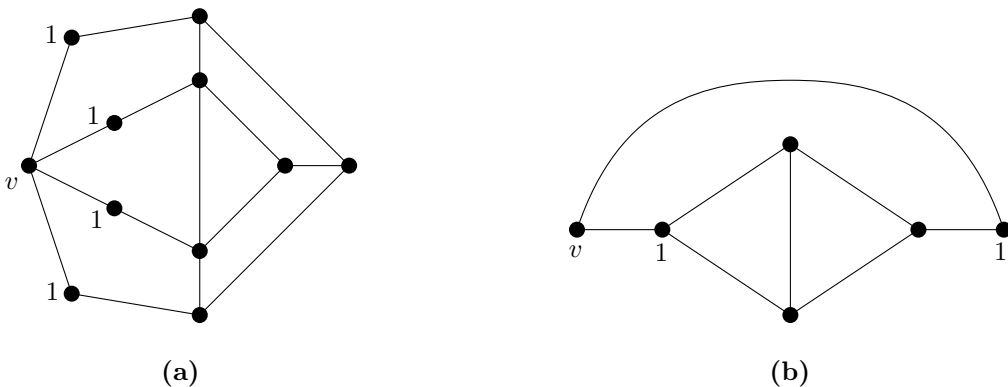**Figure 5:** Precoloring of the neighborhood of a 4-vertex v in a K_4'-free planar graph G with one triangle cannot always be extended to a 3-coloring of G (example (a)). Similarly, precoloring of the neighborhood of a 2-vertex v in a planar graph G with two triangles cannot always be extended to a 3-coloring of G (example (b)).

Note that we only consider simple graphs, i.e., loopless graphs without parallel edges; thus, whenever we perform identification of vertices in our proofs, we discard eventual parallel edges.

For a graph G, we denote the number of its vertices and edges by n_G and m_G, respectively. If G is a *plane graph*, i.e., a planar graph embedded in the plane, we denote the set of its faces by $F(G)$ and their number by f_G; in particular, the number of faces of length k is denoted by $f_{k,G}$ or simply f_k if G is evident from the context. The length of a face α is denoted by $\ell(\alpha)$. A vertex of degree k (resp., at least k) is called a *k-vertex* (resp., a *k^+-vertex*), and similarly, a face of length k is called a *k-face*.

We denote the graph obtained from a graph G by deleting a vertex v (resp., an edge e) by $G - v$ (resp., $G - e$). A graph is *k-critical*, if $\chi(G) = k$ and for any $x \in V(G) \cup E(G)$, $\chi(G - x) < k$. A subgraph of G induced by a set of vertices U is denoted by $G[U]$.

For a given cycle C in a plane embedding of a graph G, we define $\text{int}(C)$ to be the graph induced by the vertices lying strictly in the interior of C. Similarly, $\text{ext}(C)$ is the graph induced by the vertices lying strictly in the exterior of C. A *separating* cycle is a cycle C such that $\text{int}(C) \neq \emptyset$ and $\text{ext}(C) \neq \emptyset$.

The following lemma is a crucial tool in the proofs, where we use minimality of counterexamples; see, e.g., [10] for its proof.

Lemma 2.1 (Lemma 10, Borodin). *Let G be a plane graph and $F = v_1 v_2 v_3 v_4$ be a 4-face in G such that $v_1 v_3$, $v_2 v_4 \notin E(G)$. Let G_i be obtained from G by identifying v_i and v_{i+2} where $i \in \{1, 2\}$. If the number of triangles increases in both G_1 and G_2, then there exists a triangle $v_i v_{i+1} z$ for some $z \in V(G)$ and $i \in \{1, 2, 3, 4\}$. Moreover, G contains vertices x and y not in F such that $v_{i+1} z x v_{i+3}$ and $v_i z y v_{i+2}$ are paths in G (indices are modulo 4).*

In the case of planar graphs with one triangle, we can use the following simpler statement of Lemma 2.1.
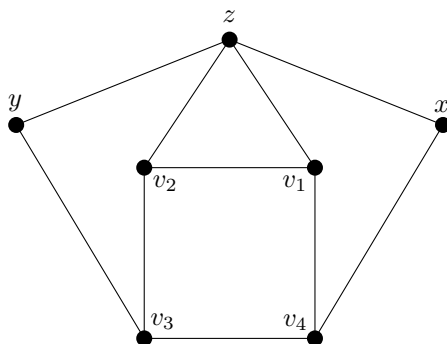
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Figure 6: The configuration in Lemma 2.1 in the case of a graph with one triangle.

Corollary 2.2. *Let G be a plane graph with at most one triangle and let α be any 4-face of G. Then, at least one of the following holds:*

(a) *α is adjacent to a triangle, or*

(b) *for at least one pair of opposite vertices of α, we can identify them without creating any new triangles.*

Theorem 1.1 settles 3-colorability of planar graphs with at most three triangles. The smallest example of a planar graph with four triangles that is not 3-colorable is the complete graph K_4. Plane 4-critical graphs with exactly four triangles have been completely characterized by Borodin et al. [8]. In their proofs, they used the following result of Kostochka and Yancey [28], which is a stronger version of Theorem 1.2.

Theorem 2.3 ([28]). *If G is a 4-critical graph, then*

$$m_G \geq \frac{5n_G - 2}{3}.$$

Moreover, the equality is achieved if and only if G is a 4-Ore graph.

Here, a graph is *k-Ore* if it is obtained from a set of copies of K_k by a sequence of DHGO-compositions, where a *DHGO-composition* $O(G_1, G_2)$ of graphs G_1 and G_2 is the graph obtained through the sequence of the following steps: delete some edge xy from G_1, split a vertex z of G_2 into non-isolated vertices z_1 and z_2, and identify x with z_1 and y with z_2.

By a $\mathrm{Pl}_{4,4f}$-*graph* we denote a planar graph with exactly four triangles and no 4-faces. A correlation between 4-Ore graphs and $\mathrm{Pl}_{4,4f}$-graphs was given in [8].

Theorem 2.4 ([8]). *Every 4-Ore graph has at least four triangles. Moreover, a 4-Ore graph has exactly four triangles if and only if it is a $\mathrm{Pl}_{4,4f}$-graph.*

3   Proofs of Theorems 1.6, 1.8, and 1.11

We prove Theorem 1.6 in two steps. First, we consider the case when the two precolored vertices receive distinct colors, which is equivalent to the statement of Theorem 3.1.

Theorem 3.1. *Let G be a planar graph with at most one triangle and let H be a graph such that $G = H - e$ for some edge e of H. Then H is 3-colorable.*

Proof. We prove the theorem by contradiction. Suppose that H is a counterexample minimizing the number of vertices plus the number of edges and let G be a plane graph with at most one triangle such that $G = H - e$ for some edge e of H. Note that since G is planar and contains at most one triangle, it is 3-colorable by Theorem 1.1. By Theorem 1.4, we may assume that G contains exactly one triangle T. Moreover, by the minimality, H is 4-critical.

We consider five cases regarding 4-faces in G.

Case 1: *G has at most two 4-faces.* By the Handshaking Lemma, we have

$$2m_G = \sum_{\alpha \in F(G)} \ell(\alpha) \geq 3 + 4 \cdot f_{4,G} + 5 \cdot (f_G - (1 + f_{4,G})) = 5f_G - 2 - f_{4,G}$$

(in the calculation, we assume that T is a face, otherwise the lower bound on the number of edges would be even higher). Then, $5f_G \leq 2m_G + 4$ and by applying the Euler's Formula and observing that $n_H = n_G$ and $m_H = m_G + 1$, we infer that

$$10 = 5n_G - 5m_G + 5f_G \leq 5n_G - 3m_G + 4 = 5n_H - 3(m_H - 1) + 4 \,.$$

Thus,

$$m_H \leq \frac{5n_H - 3}{3} \,,$$

a contradiction to Theorem 1.2.

Case 2: *G has a 4-face $\alpha = v_1v_2v_3v_4$ such that at most one vertex of α is incident with T and at most one vertex of e is incident with α.* Let G_i be the graph obtained from G by identifying v_i and v_{i+2}, where $i \in \{1, 2\}$. By the assumption and Corollary 2.2, we may assume, without loss of generality, that G_1 contains T as the unique triangle. Note that the graph H_1 obtained from H by identifying v_1 and v_3 contains e and is thus 3-colorable by the minimality. Thus, we can extend the coloring of H_1 to the coloring of H in which v_1 and v_3 receive the same color, a contradiction.

Case 3: *G has a 4-face $\alpha = v_1v_2v_3v_4$ such that at most one vertex of α is incident with T and both vertices of e are incident with α.* We may assume, without loss of generality, that $e = v_1v_3$. Let G_2 be the graph obtained from G by identifying v_2 and v_4. Note that if the number of triangles does not increase in G_2, then we can continue as in Case 2.

Therefore, by Lemma 2.1, there exist vertices $x, z \in V(G)$ such that $xv_4, xz, zv_2 \in E(G)$. Consequently, no 4-face of G, other than α, can contain both vertices v_1 and v_3 due to planarity.

Due to Cases 1 and 2, and the fact that α contains both vertices of e, there exists a 4-face $\alpha' = v_1'v_2'v_3'v_4'$ such that α' contains two vertices of T, say v_1' and v_2' (note that the two vertices incident with T are not opposite in α', otherwise there would be another triangle in G), with z' being the third vertex of T. Let G_i' be the graph obtained from G by identifying v_i' and v_{i+2}', where $i \in \{1, 2\}$. Again, if the number of triangles does not increase in G_1' or G_2', then we can color H with 3-colors.

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It follows that there exist vertices $x', y' \in V(G)$ such that $x'z'$, $x'v_4'$, $y'z'$, and $y'v_3' \in E(G)$. Suppose that at least one of $C_1 = z'v_2'v_3'y'$ or $C_2 = z'v_1'v_4'x'$ is a 4-face, say C_1. By our observation above, C_1 does not contain both vertices of e. Let G' be the graph obtained from G by identifying v_2' and y'. Note that the number of triangles in G' does not increase. Let H' be the graph obtained from G' by adding the edge e. By the minimality, we can color H' with 3 colors and extend the coloring to a coloring of H, in which y' and v_2' receive the same color, a contradiction.

Thus, we may assume that both C_1 and C_2 are separating 4-cycles. Note that if the vertices of α (and thus also the endvertices of e) belong to the vertex set $V_1 = V(\text{ext}(C_1)) \cup V(C_1)$, then $H[V_1]$ contains both T and e. Therefore, we can color $H[V_1]$ by the minimality and extend the coloring of C_1 to a coloring of $H[V(\text{int}(C_1)) \cup V(C_1)]$ by Theorem 1.5. We use an analogous argument for C_2 in the case when the vertices of α belong to the graph induced by the vertex set $V(\text{int}(C_1)) \cup V(C_1)$, which implies that the vertices of α belong to the vertex set $V(\text{ext}(C_2)) \cup V(C_2)$. Thus, H is 3-colorable, a contradiction.

Case 4: G has a 4-face $\alpha = v_1v_2v_3v_4$ such that exactly two of its vertices, say v_1 and v_2, are incident with T, and at most one vertex of the edge e is incident with α. Let z be the third vertex of T. Using similar arguments as in the previous cases, we infer that there exist vertices $x, y \in V(G)$ such that xz, xv_4, yz, and $yv_3 \in E(G)$.

Suppose that $C_1 = zv_2v_3y$ is a 4-face. If $e \neq v_2y$, then consider the graph G' obtained from G by identifying v_2 and y. Note that the number of triangles in G' does not increase. Let H' be the graph obtained from G' by adding the edge e. By the minimality, we can color H' with 3-colors and extend the coloring to a coloring of H, in which y and v_2 receive the same color, a contradiction. Therefore, $e = v_2y$. But then, either $C_2 = zv_1v_4x$ is a 4-face, in which case we can apply the same procedure on v_1 and x as we did on v_2 and y, or C_2 is a separating 4-cycle. However, since both $V(T)$ and $V(e)$ belong to the vertex set $V_1 = V(\text{ext}(C_2)) \cup V(C_2)$, we can complete the coloring in a similar manner as in the last paragraph of Case 3, a contradiction.

Thus, by symmetry, both C_1 and C_2 are separating 4-cycles. Moreover, each of C_1 and C_2 contains exactly one vertex of e in its interior. Furthermore, T is a 3-face, otherwise we can color $H[V(\text{ext}(T)) \cup V(T)]$ by the minimality, and then extend the coloring to the interior of T by Theorem 1.5. Additionally, due to Case 1, there exists a 4-face $\alpha' = v_1'v_2'v_3'v_4'$ in G, distinct from α. If identifying either v_1' and v_3', or v_2' and v_4' results in a graph with one triangle, namely T, then by the minimality, it is 3-colorable and the coloring can be extended to H. Therefore, by the fact that G has only one triangle and Lemma 2.1, two vertices of α' are incident with T, say $v_1' = v_1$ and $v_2' = z$ (meaning that at least one of v_3' and v_4' is in $V(\text{int}(C_2))$, see Figure 7) and there are vertices x' and y' in G such that $x'v_4'$, $x'v_2$, $y'v_3'$, and $y'v_2 \in E(G)$. This is not possible due to the planarity of G, a contradiction.

Case 5: G has at least three 4-faces and each of them is incident with two vertices of T and both vertices of e. Let $\alpha = v_1v_2v_3v_4$ be such a face and let $T = v_1v_2z$. Without loss of generality, we may assume that $e = v_1v_3$. Let G_2 be the graph obtained from G by identifying v_2 and v_4. If the number of triangles does not increase in G_2, then we are
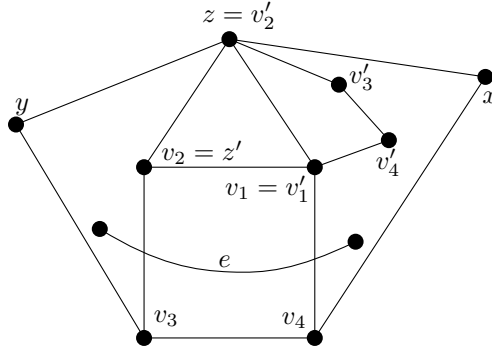
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Figure 7: The 4-faces α and α' in the last part of Case 4.

done. Thus, by Lemma 2.1, there exists a vertex $x \in V(G)$ such that xz, $xv_4 \in E(G)$. Note that by the assumptions, $C = zv_1v_4x$ is not a 4-face, since it is incident to exactly one vertex of e. Therefore, C is a separating 4-cycle. But then, the vertices of both T and e are contained in the vertex set $V_1 = V(\text{ext}(C)) \cup V(C)$. Let $V_2 = V(\text{int}(C)) \cup V(C)$. By the minimality, we can color $G[V_1]$ and extend the coloring of C to the coloring of $G[V_2]$ by Theorem 1.5, a contradiction.

Since no 4-face can be incident with all three vertices of T, the proof is completed. $\square$

In the second step of proving Theorem 1.6, we show that any two non-adjacent vertices in a planar graph with one triangle can be colored with the same color.

Theorem 3.2. *Let G be a planar graph with at most one triangle. Then each coloring of any two non-adjacent vertices with the same color can be extended to a 3-coloring of G.*

Proof. We prove the theorem by contradiction. Suppose that a counterexample G is a plane graph with the minimum number of vertices. By Theorem 1.4, we may also assume that G contains exactly one triangle T. Let u and v be two non-adjacent vertices of G.

Let H be the graph obtained from G by identifying the vertices u and v. Clearly, $n_G = n_H + 1$ and $m_G = m_H$. By the minimality, H is 4-critical. To reach a contradiction, we only need to prove that H is 3-colorable, which implies that there exists a 3-coloring of G in which u and v receive the same color.

We consider three cases regarding 4-faces in G.

Case 1: *G has no 4-faces.*   By the Handshaking Lemma, we have

$$2m_G = \sum_{\alpha \in F(G)} \ell(\alpha) \geq 3 + 5 \cdot (f_G - 1) = 5f_G - 2 \,.$$

Then, $5f_G \leq 2m_G + 2$ and by applying the Euler's Formula, we infer that

$$10 = 5n_G - 5m_G + 5f_G \leq 5n_G - 3m_G + 2 = 5n_H + 5 - 3m_H + 2 \,.$$

Thus,

$$m_H \leq \frac{5n_H - 3}{3} \,,$$

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a contradiction to Theorem 1.2.

Case 2: *G has exactly one 4-face.* Similarly as in Case 1, we can compute that $5f_G \leq 2m_G + 3$ and by applying Euler's Formula, we infer that

$$m_H \leq \frac{5n_H - 2}{3}\,.$$

In the case when $m_H < \frac{5n_H - 2}{3}$, we obtain a contradiction to Theorem 1.2, and therefore, H has exactly $\frac{5n_H - 2}{3}$ edges.

Let $\alpha = v_1 v_2 v_3 v_4$ be the 4-face in G and let G_i be the graph obtained from G by identifying v_i and v_{i+2}, where $i \in \{1, 2\}$.

Suppose first that the number of triangles does not increase in G_1 or G_2, say G_1. In the case $\{u, v\} \neq \{v_1, v_3\}$, we identify v_1 and v_3 in H to obtain the graph H_1. By the minimality, we can color H_1 with 3 colors and extend the coloring to a coloring of H, and therefore also to G, a contradiction. Hence, we may assume that $\{u, v\} = \{v_1, v_3\}$. In this case, H is a planar graph with exactly one triangle. Thus, by Theorem 1.1, there exists a 3-coloring of H, and therefore also of G, a contradiction.

We may thus assume that the number of triangles increases in both G_1 and G_2. By Lemma 2.1, without loss of generality, we may assume that there exist vertices $x, y, z \in V(G)$ such that zv_1, zv_2, xz, xv_4, yz, and $yv_3 \in E(G)$, where zv_1v_2 is T. Since G contains exactly one 4-face, it follows that both $C_1 = zv_1v_4x$ and $C_2 = zv_2v_3y$ are separating 4-cycles.

Note that if both u and v belong to the subgraph of G induced by the vertex set $V_1 = V(\text{ext}(C_1)) \cup V(C_1)$, then we can color $G[V_1]$ by the minimality and use Theorem 1.5 to extend the coloring of C_1 to the coloring of the interior of C_1. By symmetry, we may thus assume, without loss of generality, that $u \in V(\text{int}(C_1))$ and $v \in V(\text{int}(C_2))$.

Since $m_H = \frac{5n-2}{3}$, by Theorems 2.3 and 2.4, we infer that H must have at least 5 triangles. Therefore, since G has exactly one triangle, it follows that by identifying u and v, we create at least four new triangles. We will prove that this cannot happen.

First, observe that no new triangle can contain vertices x or y, since that would imply the existence of another triangle, distinct from T, in G. Next, observe that u is adjacent with at most one of the vertices v_1 and v_4, and v is adjacent with at most one of the vertices v_2 and v_3. Thus, at most one new triangle can be formed using the edges v_1v_2 or v_3v_4, and so there must exist at least three triangles in H which contain the vertex z and either u or v, say u, is adjacent to z. Therefore, there exist at least three vertices $w_1, w_2, w_3 \in V(G)$ such that $w_1, w_2, w_3 \in V(\text{int}(C_2))$. Moreover, each of them is adjacent to z and v (see Figure 8). Consider now the 4-cycle $C = zw_1vw_2$. Since G contains exactly one 4-face, it follows that C is a separating 4-cycle. Furthermore, the exterior of C together with the vertices of C contains both u and v, as well as T. Thus, by the minimality, we can color $G[V(\text{ext}(C)) \cup V(C)]$ and extend the 3-coloring of the vertices of C to a 3-coloring of H by Theorem 1.5.

Case 3: *G has at least two 4-faces.* Let $\alpha = v_1 v_2 v_3 v_4$ be a 4-face and let G_i be the graph obtained from G by identifying v_i and v_{i+2}, where $i \in \{1, 2\}$. Since G contains exactly one triangle, by Corollary 2.2, either, without loss of generality, v_1v_2 is an edge of
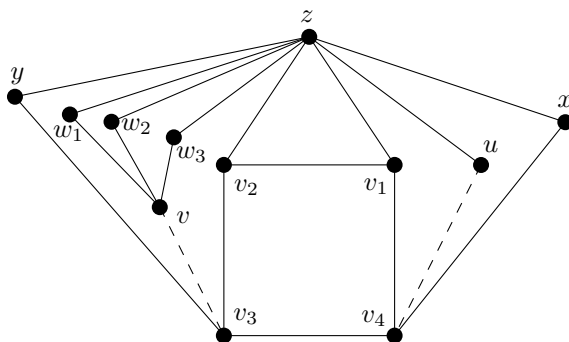
Figure 8: The vertices in G comprising triangles in H in the last part of Case 2.

T or we can identify v_1 and v_3 or v_2 and v_4 without creating any new triangles. Suppose first that v_1v_2 is not an edge of T; say that G_1 has at most one triangle. Then, in the case $\{u, v\} \neq \{v_1, v_3\}$, we identify v_1 and v_3 in H to obtain the graph H_1. By the minimality, we can color H_1 with 3 colors and extend the coloring to a coloring of H, and therefore also to G, a contradiction. Hence, we may assume that $\{u, v\} = \{v_1, v_3\}$. In this case, H is a planar graph with exactly one triangle. Thus, by Theorem 1.1, there exists a 3-coloring of H, and therefore also of G, a contradiction.

Thus, we may assume that $T = v_1v_2z$, with z being distinct from v_3 and v_4, and that in both G_1 and G_2 the number of triangles is at least 2. Therefore, by Lemma 2.1, there exist vertices $x, y \in V(G)$ such that xz, xv_4, yz, and $yv_3 \in E(G)$.

Suppose that $C_1 = zv_1v_4x$ is a 4-face. Then, due to planarity of G, in the graph G' obtained by identifying v_1 and z, no new triangle is created. Thus, by the minimality, we can color G' and infer 3-colorability of G in a similar manner as above, a contradiction.

Therefore, by symmetry, we may assume that both C_1 and $C_2 = zv_2v_3y$ are separating 4-cycles. Note that if both u and v belong to the vertex set $V_1 = V(\text{ext}(C_1)) \cup V(C_1)$ (resp., $V_2 = V(\text{ext}(C_2)) \cup V(C_2)$), then, by the minimality, we can color the graph H_1 (resp., H_2) obtained from $G[V_1]$ (resp., $G[V_2]$) by identifying u and v and extend the coloring to a coloring of H by Theorem 1.5, hence also obtaining a 3-colorability of G.

Thus, we may assume, without loss of generality, that $u \in \text{int}(C_1)$ and $v \in \text{int}(C_2)$. Now, consider a 4-face $\alpha' = v_1'v_2'v_3'v_4'$. If α' satisfies the property (b) of Corollary 2.2, then we proceed as above to obtain a contradiction. Therefore, α' is incident with T and, by planarity of G, the vertices of α' are all contained in $V(\text{int}(T)) \cup V(T)$. But then, both u and v belong to the exterior of T and we can color, by the minimality, the graph obtained from $G[V(\text{ext}(T) \cup V(T))]$ by identifying u and v. Finally, we extend the obtained coloring to a coloring of H by Theorem 1.5. Hence, from the coloring of H, we again obtain 3-colorability of G, a contradiction. This completes the proof. $\qquad\square$

Theorems 3.1 and 3.2 combined settle Theorem 1.6. Next, we prove Theorem 1.8.

Proof of Theorem 1.8. Let G be a planar graph with at most one triangle and let f be a precolored face of length at most 4.

Suppose first that f is of length 3. Since there is only one coloring of f (up to a permutation of colors), the result follows from Theorem 1.1.

Thus, we may assume that $f = v_1 v_2 v_3 v_4$ is a 4-face. Suppose that the precoloring of f uses all three colors. Then, two non-adjacent vertices of f, say v_1 and v_3, receive distinct colors and the other two vertices are colored with the third. Note that the same coloring of f (up to a permutation of colors) can be obtained by adding an edge between v_1 and v_3. The obtained graph is 3-colorable by Theorem 3.1.

Therefore, we may assume that the vertices of f are precolored with two colors. We proceed by contradiction. Let G be a plane graph with at most one triangle such that a precoloring of some 4-face f with two colors cannot be extended to a 3-coloring of G. Moreover, let G be the smallest such graph in terms of the vertices. Clearly, G has exactly one triangle T, otherwise the precoloring can be extended by Theorem 1.5.

Let G_i be the graph obtained from G by identifying v_i and v_{i+2}, where $i \in \{1, 2\}$. If the number of triangles does not increase in G_1 or G_2, say G_1, then there is a 3-coloring of G_1, guaranteed by Theorem 1.6, which induces a 3-coloring of G such that the vertices of f are colored with two colors.

Thus, by Lemma 2.1, without loss of generality, we may assume that there exist vertices $x, y, z \in V(G)$ such that zv_1, zv_2, xz, xv_4, yz, and $yv_3 \in E(G)$, where $T = zv_1v_2$. Observe that coloring of f forces also the colors on x, y, and z (see Figure 9).
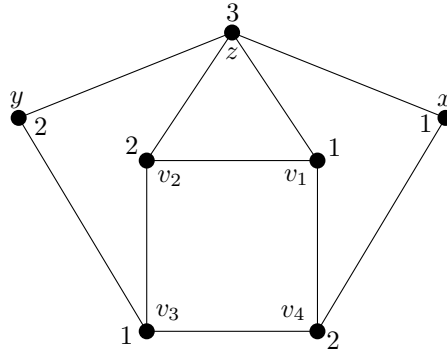

Figure 9: The coloring of f forces the colors of x, y, and z.

Let $C_1 = zv_1v_4x$ and $C_2 = zv_2v_3y$. Suppose that at least one of C_1 or C_2, say C_1, is a separating 4-cycle. Then, by the minimality, the coloring of f extends to a 3-coloring of $G[V(\text{ext}(C_1)) \cup V(C_1)]$. Since the obtained coloring of C_1 extends to a 3-coloring of $G[V(\text{int}(C_1)) \cup V(C_1)]$ by Theorem 1.5, we obtain a 3-coloring of G, a contradiction.

Thus, we may assume that both C_1 and C_2 are 4-faces in G. In a similar manner as above, we infer that T must be a 3-face. But then, the precoloring of the 5-cycle $C_3 = v_3v_4xzy$ given in Figure 9 extends to a 3-coloring of $G[V(\text{ext}(C_3) \cup V(C_3))]$ (which might as well be an empty graph) by Theorem 1.5 and we color the two vertices in the interior of C_3 as in Figure 9, hence obtaining a 3-coloring of G, a contradiction. This completes the proof. $\qquad\square$

We conclude this section with a proof of Theorem 1.11.

Proof of Theorem 1.11. We prove the theorem by contradiction. Let G be a minimal counterexample to the theorem, i.e., G is a K_4'-free planar graph with at most one triangle and the minimum number of vertices such that there is a vertex u of degree at most 3 with an independent neighborhood, such that precoloring the vertices in $N(u)$ with a same color does not extend to a 3-coloring of G.

First, observe that by Theorem 1.3, G has exactly one triangle T, and by Theorem 1.6, u is a 3-vertex. Let $N[u] = \{u, u_1, u_2, u_3\}$ and let H be the graph obtained by identifying $N[u]$ into a vertex w. Let α_1, α_2, and α_3 be the three faces incident to u in G that contain respectively $\{u_1, u_2\}$, $\{u_2, u_3\}$, and $\{u_1, u_3\}$. Furthermore, let α_1', α_2', and α_3' be the faces incident to w in H corresponding to α_1, α_2, and α_3.

Clearly, every 3-coloring of H induces a 3-coloring of G with u_1, u_2, and u_3 colored with a same color, while u can be colored with either of the remaining two colors. Additionally, since G is a planar graph, H is also a planar graph and by the minimality of G, H is 4-critical. Observe also that $n_G = n_H + 3$, $m_G = m_H + 3$, and $f_G = f_H$.

Now, we prove two structural properties of H.

Claim 1. *H has no separating triangles.*

Proof. Suppose the contrary and let C be a separating triangle in H. First, suppose that C is the triangle of G. Without loss of generality, we may assume that $w \in V(\text{int}(C))$. By the minimality, there is a 3-coloring of $H[V(\text{int}(C)) \cup V(C)]$, and by Theorem 1.1, we can extend it to a 3-coloring of H, since $H[V(\text{ext}(C)) \cup V(C)]$ has exactly one triangle, a contradiction.

Therefore, we may assume that $C \neq T$. In that case, C has been created from a 5-cycle C_G after we identified $N[u]$ into w and thus $w \in V(C)$. Since $C \neq T$, we may assume, without loss of generality, that $H[V(\text{int}(C)) \cup V(C)]$ contains α_1' but not α_2' or α_3' (see Figure 10).
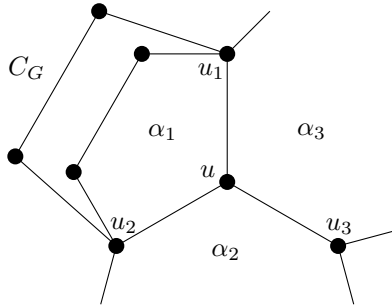

Figure 10: A separating 5-cycle in G containing α_1.

By the minimality, there is a 3-coloring ϕ of $H[V(\text{ext}(C)) \cup V(C)]$. Now, we show that we can extend ϕ to the interior of C. Let $H_1 = H[V(\text{int}(C)) \cup V(C)]$. We proceed by induction on the number of separating triangles in H_1. First, recall that all separating triangles in H_1 are incident to w; more precisely, they were obtained from 5-cycles in G containing $\{u_1, x, u_2\}$.

Suppose that $H_1 = H[V(\text{int}(C)) \cup V(C)]$ has no separating triangle. Then it has at most three triangles: C as its outer face, possibly α'_1, and possibly T. Therefore, H_1 is a planar graph with at most three triangles and thus 3-colorable by Theorem 1.1.

So, we may assume that H_1 has at least one separating triangle; we select a separating triangle C' such that all separating triangles in H_1 are contained in $H'_1 = H_1[V(\text{int}(C')) \cup V(C')]$. Then, by induction, there is a 3-coloring ϕ' of H'_1. Finally, using the colorings ϕ and ϕ', we can complete the coloring of H by coloring $H[V(H_1) \setminus V(\text{int}(C'))]$ using Theorem 1.1 and an eventual permutation of colors in ϕ', a contradiction. $\blacklozenge$

Claim 2. *If H has a separating 4-cycle, then both its interior and exterior must contain w or a triangle.*

Proof. Suppose the contrary and let C be a separating 4-cycle of H such that $H[V(\text{int}(C)) \cup V(C)]$ is a triangle-free planar graph that does not contain w. By the minimality, there is a 3-coloring ϕ of $H[V(\text{ext}(C)) \cup V(C)]$. By Theorem 1.5, we can extend ϕ to the whole graph H, a contradiction. $\blacklozenge$

Now, we are ready to finish the proof by considering three cases regarding 4-faces of G.

Case 1: *G has no 4-faces.* By the Handshaking Lemma, we have $2m_G \geq 5f_G - 2$ and so $2m_H + 6 \geq 5f_H - 2$. Then, $5f_H \leq 2m_H + 8$ and by applying the Euler's Formula on G, we infer that

$$m_H \leq \frac{5n_H - 2}{3} \ .$$

Since H is 4-critical, by Theorem 2.3, we have that $m_H = \frac{5n_H - 2}{3}$ and that H is a 4-Ore graph. Moreover, since H does not have separating triangles by Claim 1, there are at most four triangles in H (T and the faces α'_1, α'_2, and α'_3). Thus, by Theorem 1.1, H has exactly four triangles and by Theorem 2.4, H is a $\text{Pl}_{4,4f}$-graph. Recall that three of the triangles are incident to the same vertex w. The only $\text{Pl}_{4,4f}$-graph for which this is true is K_4 [8, Theorem 4]. However, to obtain K_4, all three neighbors of u in G must be of degree 2, meaning that G must be K'_4, a contradiction.

Case 2: *G has a 4-face that is incident to u.* As a result, after identifying u and its neighbors, H has at most three triangles by Claim 1. Therefore, H is 3-colorable by Theorem 1.1.

Case 3: *G has a 4-face $\alpha = v_1 v_2 v_3 v_4$ and α is not incident to u.* The edges $v_1 v_3$ and $v_2 v_4$ are not present in G, otherwise G would have at least two triangles. Moreover, if u is adjacent to two (opposite) vertices of α, say $v_1 = u_1$ and $v_3 = u_3$, then, by Case 2, neither $C_1 = u v_1 v_2 v_3$ nor $C_2 = u v_1 v_4 v_3$ is a 4-face. Moreover, without loss of generality, we may assume that $u_2 \in V(\text{ext}(C_1))$. However, by the minimality, there is a 3-coloring of $G[V(\text{ext}(C_1)) \cup V(C_1)]$, and it can easily be extended to the whole G by Theorem 1.8. Therefore, at most one of the vertices of α is adjacent to u.

Let G_i be the graph obtained from G by identifying v_i and v_{i+2}, where $i \in \{1,2\}$. Suppose that the only triangle in G_1 is T. Then, by the minimality, the graph H_1

obtained by identifying the vertices v_1 and v_3 in H is 3-colorable. Clearly, its coloring can be extended to H and thus also to G, a contradiction.

Therefore, by symmetry, we may assume that in G_1 and G_2 the number of triangles increases. It follows by Lemma 2.1 that there are vertices x, y, and z such that $v_1z, v_2z, xz, xv_4, yz, yv_3 \in E(G)$, where $T = v_1v_2z$. If one of $C_1 = zv_1v_4x$ and $C_2 = zv_2v_3y$ is a 4-face, it has the same properties as α and two of its vertices are incident with T. But that is not possible due to planarity.

Thus, C_1 and C_2 are separating 4-cycles of G. Since, at most one of them can contain $u \neq z$ (by definition, u is not incident with a triangle), the other one remains a separating 4-cycle of H, which does not contain T nor w, a contradiction to Claim 2. This completes the proof. $\qquad\square$

4   Conclusion

One motivation for the research presented in this paper was a conjecture on adynamic coloring of planar graphs with one triangle. An *adynamic coloring* is a proper vertex coloring of a graph G such that for at least one 2^+-vertex all of its neighbors are colored with a same color. Clearly, to admit such a coloring, G must have at least one 2^+vertex v with an independent neighborhood, i.e., v is not incident to a triangle. This is also a sufficient condition.

In [30], it was proved that every triangle-free planar graph admits an adynamic 3-coloring (note that this fact is also a corollary of Theorem 1.3). On the other hand, there are planar graphs with two triangles that need 4 colors (see, e.g., the graph in Figure 5(b)). Regarding planar graphs with one triangle, Šurimová et al. [30] conjectured that they are 3-colorable as soon as they contain a 2^+-vertex with an independent neighborhood. Using the results of this paper, we are able to answer the conjecture in affirmative.

Theorem 4.1. *Every planar graph with at most one triangle and a 2^+-vertex with an independent neighborhood is adynamically 3-colorable.*

Proof. We again proceed by contradiction. Let G be a minimum counterexample in terms of the number of vertices with some fixed embedding. By Theorem 1.3, G has exactly one triangle T. Suppose first that there is a 2-vertex v in G and let $N(v) = \{v_1, v_2\}$. The graph G' obtained by splitting v into two adjacent vertices both connected to v_1 and v_2 is planar with at most three triangles and thus 3-colorable by Theorem 1.1. Its coloring induces a coloring of G in which v_1 and v_2 receive the same color, a contradiction.

Therefore, $\delta(G) \geq 3$. Moreover, by the Handshaking Lemma and the Euler's Formula, there are at least nine 3-vertices in G, and so at least six 3-vertices are not incident with T. Hence, by Theorem 1.11, G contains a subgraph D isomorphic to K_4'. Since $\delta(G) \geq 3$, there is a 5-face α of D that is not a face in G.

The graph induced by the interior of α in G and $V(\alpha)$ is a triangle-free plane graph, which we can 3-color adynamically. This coloring gives us a coloring of the vertices of α and fixes also the color of the vertex of T that is not incident with α. It remains to color (eventual) interiors of the other two 5-faces of D in G and the interior of T. All can be colored by Theorem 1.5. This completes the proof. $\qquad\square$

Regarding other results in this paper, there are a number of possibilities for further work. For example, similar results for planar graphs with two triangles and three triangles would be interesing.

Problem 4.2. *Characterize planar graphs with two (resp., three) triangles, in which pre-coloring of any two non-adjacent vertices extends to a 3-coloring of the graph.*

Also, one could investigate in more details precoloring extensions from larger independent sets.

Problem 4.3. *Characterize planar graphs with one triangle, in which precoloring of any three non-adjacent vertices extends to a 3-coloring of the graph.*

Additionally, Problem 4.3 could be extended to determining the properties of triples of non-adjacent vertices whose precoloring does not extend to the whole graph; particularly, which colorings of them.

We showed that a precoloring of a 5-face in a planar graph G with one triangle cannot always be extended to a 3-coloring of G. So it is natural to ask for a characterization similar to characterizations for faces of lengths 6 to 9 in triangle-free planar graphs.

Problem 4.4. *Characterize planar graphs with one triangle, in which precoloring of a 5-face (resp., k-face for any $k \geq 6$) extends to a 3-coloring of the graph.*

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