2-Distance list (Δ+2)-coloring of planar graphs with girth at least 10

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2-distance list $(\Delta + 2)$-coloring of planar graphs with girth at least 10

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Abstract

Given a graph G and a list assignment $L(v)$ for each vertex of v of G. A proper L-list-coloring of G is a function that maps every vertex to a color in $L(v)$ such that no pair of adjacent vertices have the same color. We say that a graph is list k-colorable when every vertex v has a list of colors of size at least k. A 2-distance coloring is a coloring where vertices at distance at most 2 cannot share the same color. We prove the existence of a 2-distance list $(\Delta + 2)$-coloring for planar graphs with girth at least 10 and maximum degree $\Delta \geq 4$.

1 Introduction

A *k-coloring* of the vertices of a graph $G = (V, E)$ is a map $\phi : V \to \{1, 2, \ldots, k\}$. A k-coloring ϕ is a *proper coloring*, if and only if, for all edge $xy \in E, \phi(x) \neq \phi(y)$. In other words, no two adjacent vertices share the same color. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k such that G has a proper k-coloring. A generalization of k-coloring is k-list-coloring. A graph G is *L-list colorable* if for a given list assignment $L = \{L(v) : v \in V(G)\}$ there is a proper coloring ϕ of G such that for all $v \in V(G), \phi(v) \in L(v)$. If G is L-list colorable for every list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$, then G is said to be *k-choosable* or *k-list-colorable*. The *list chromatic number* of a graph G is the smallest integer k such that G is k-choosable. List coloring can be very different from usual coloring as there exist graphs with a small chromatic number and an arbitrarily large list chromatic number.

In 1969, Kramer and Kramer introduced the notion of 2-distance coloring [19, 20]. This notion generalizes the "proper" constraint (that does not allow two adjacent vertices to have the same color) in the following way: a 2-*distance k-coloring* is such that no pair of vertices at distance at most 2 have the same color. The 2-*distance chromatic number* of G, denoted by $\chi^2(G)$, is the smallest integer k such that G has a 2-distance k-coloring. Similarly to proper k-list-coloring, one can also define 2-*distance k-list-coloring* We denote $\chi^2_l(G)$ the 2-*distance list chromatic number* of G.

For all $v \in V$, we denote $d_G(v)$ the degree of v in G and by $\Delta(G) = \max_{v \in V} d_G(v)$ the maximum degree of a graph G. For brevity, when it is clear from the context, we will use Δ (resp. $d(v)$) instead of $\Delta(G)$ (resp. $d_G(v)$). One can observe that, for any graph G, $\Delta + 1 \leq \chi^2(G) \leq \Delta^2 + 1$. The lower bound is trivial since, in a 2-distance coloring, every neighbor of a vertex v with degree Δ, and v itself must have a different color. As for the upper bound, a greedy algorithm shows that $\chi^2(G) \leq \Delta^2 + 1$. Moreover, that upper bound is tight for some graphs, for example, Moore graphs of type $(\Delta, 2)$, which are graphs where all vertices have degree Δ, are at distance at most two from each other, and the total number of vertices is $\Delta^2 + 1$. See Figure 1.

By nature, 2-distance list colorings and the 2-distance list chromatic number of a graph depend a lot on the number of vertices in the neighborhood of every vertex. More precisely, the "sparser" a graph is, the lower its 2-distance chromatic number will be. One way to quantify the sparsity of a graph is through its maximum average degree. The *average degree* ad of a graph $G = (V, E)$ is defined by $\mathrm{ad}(G) = \frac{2|E|}{|V|}$. The *maximum average degree* $\mathrm{mad}(G)$ is the maximum, over all subgraphs H of G, of $\mathrm{ad}(H)$. Another way to measure the sparsity is through the girth, i.e. the length of a shortest cycle. We denote $g(G)$ the girth of G. Intuitively, the higher the girth of a graph is, the sparser it gets. These two measures can actually be linked directly in the case of planar graphs.

A graph is *planar* if one can draw its vertices with points on the plane, and edges with curves intersecting only at its endpoints. When G is a planar graph, Wegner conjectured in 1977 that $\chi^2(G)$ becomes linear in $\Delta(G)$:

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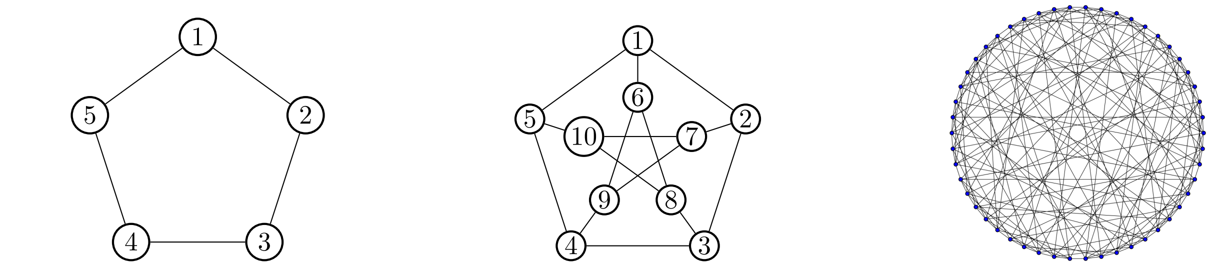


(i) The Moore graph of type (2,2): the odd cycle C_5.

(ii) The Moore graph of type (3,2): the Petersen graph.

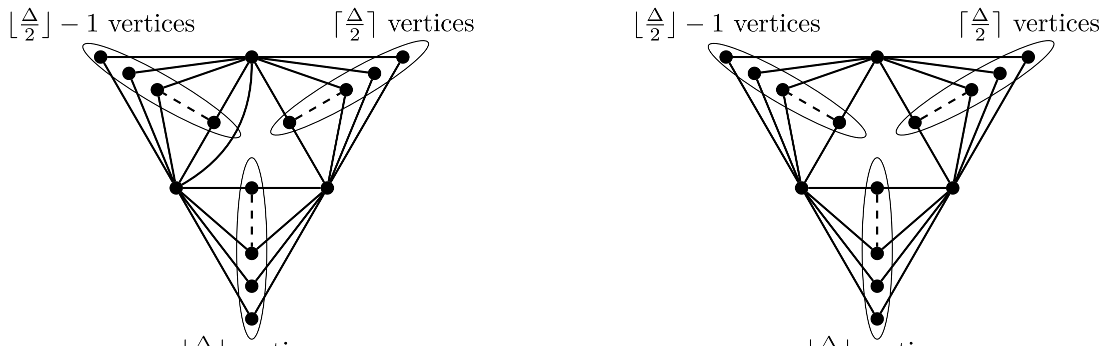
(iii) The Moore graph of type (7,2): the Hoffman-Singleton graph.

Figure 1: Examples of Moore graphs for which $\chi^2 = \Delta^2 + 1$.

Conjecture 1 (Wegner [28])**.** *Let G be a planar graph with maximum degree Δ. Then,*

$$\chi^2(G) \leq \begin{cases} 7, & \text{if } \Delta \leq 3, \\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\ \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

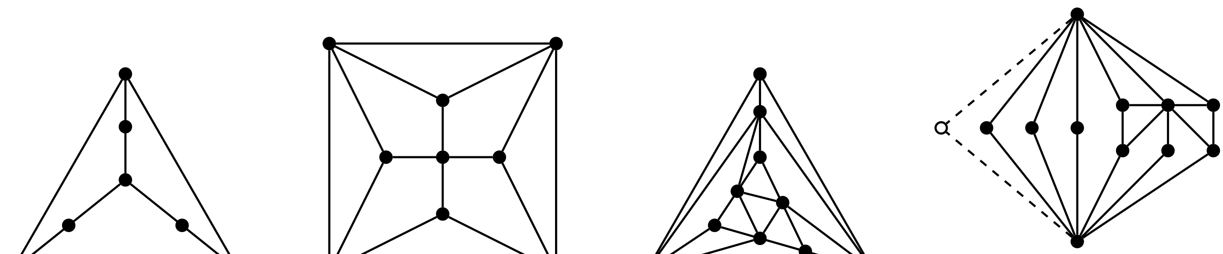
The upper bound for the case where $\Delta \geq 8$ is tight (see Figure 2(i)). Recently, the case $\Delta \leq 3$ was proved by Thomassen [27], and by Hartke *et al.* [16] independently. For $\Delta \geq 8$, Havet *et al.* [17] proved that the bound is $\frac{3}{2}\Delta(1 + o(1))$, where $o(1)$ is as $\Delta \to \infty$ (this bound holds for 2-distance list-colorings). Conjecture 1 is known to be true for some subfamilies of planar graphs, for example K_4-minor free graphs [26].



(i) A graph with girth 3 and $\chi^2 = \lfloor \frac{3\Delta}{2} \rfloor + 1$.

(ii) A graph with girth 4 and $\chi^2 = \lfloor \frac{3\Delta}{2} \rfloor - 1$.

Figure 2: Graphs with $\chi^2 \approx \frac{3}{2}\Delta$ [28].



(i) $\Delta = 3$ and $\chi^2 \geq 7$.

(ii) $\Delta = 4$ and $\chi^2 \geq 9$.

(iii) $\Delta = 5$ and $\chi^2 \geq 10$.

(iv) $6 \leq \Delta \leq 7$ and $\chi^2 \geq \Delta + 5$.

Figure 3: Constructions by Wegner in [28].

Wegner's conjecture motivated extensive researches on 2-distance chromatic number of sparse graphs, either of planar graphs with high girth or of graphs with upper bounded maximum average degree which are directly linked due to Proposition 2.

Proposition 2 (Folklore)**.** *For every planar graph G,* $(\text{mad}(G) - 2)(g(G) - 2) < 4$.

As a consequence, any theorem with an upper bound on $\mathrm{mad}(G)$ can be translated to a theorem with a lower bound on $g(G)$ under the condition that G is planar. Many results have taken the following form: *every graph G of $\mathrm{mad}(G) \leq m_0$ and $\Delta(G) \geq \Delta_0$ satisfies $\chi^2(G) \leq \Delta(G) + c(m_0, \Delta_0)$ where $c(m_0, \Delta_0)$ is a small constant depending only on m_0 and Δ_0*. Due to Proposition 2, as a corollary, we have the same results on planar graphs of girth $g \geq g_0(m_0)$ where g_0 depends on m_0. Table 1 shows all known such results, up to our knowledge, on the 2-distance chromatic number of planar graphs with fixed girth, either proven directly for planar graphs with high girth or came as a corollary of a result on graphs with bounded maximum average degree.

g_0 \ $\chi^2(G)$	$\Delta+1$	$\Delta+2$	$\Delta+3$	$\Delta+4$	$\Delta+5$	$\Delta+6$	$\Delta+7$	$\Delta+8$
3				$\Delta = 3$ [27, 16] $\cancel{\Delta \geq 4}$[1] [2]	$\cancel{\Delta \geq 10}$[2]	$\cancel{\Delta \geq 12}$[2]	$\cancel{\Delta \geq 14}$[2]	$\cancel{\Delta \geq 16}$[2]
4			$\cancel{\Delta \geq 10}$[3]	$\cancel{\Delta \geq 12}$[3]	$\cancel{\Delta \geq 14}$[3]	$\cancel{\Delta \geq 16}$[3]	$\cancel{\Delta \geq 18}$[3]	$\cancel{\Delta \geq 20}$[3]
5		$\Delta \geq 10^7$ [1][6] $\cancel{\Delta = 4}$[4]	$\Delta \geq 339$ [14]	$\Delta \geq 312$ [13]	$\Delta \geq 15$ [8][5]	$\Delta \geq 12$ [7][6]	$\Delta \neq 7, 8$ [13]	all Δ [12]
6		$\Delta \geq 17$ [3][9]	$\Delta \geq 9$ [7][6]		all Δ [9]			
7	$\Delta \geq 16$ [18][6]	$\Delta \geq 10$ [24]	$\Delta \geq 6$ [21][7]	$\Delta = 4$ [10][7]				
8	$\Delta \geq 9$ [25][5]	$\Delta \geq 6$ [24]	$\Delta \geq 4$ [21][7]					
9	$\Delta \geq 7$ [23][9]	$\Delta = 5$ [6][7]	$\Delta = 3$ [11][6]					
10	$\Delta \geq 6$ [18][6]	$\Delta \geq 4$[8]						
11	$\cancel{\Delta = 3}$ [22]	$\Delta = 4$ [10][7]						
12	$\Delta = 5$ [18][6]	$\Delta = 3$ [5][6]						
13								
14	$\Delta \geq 4$ [2][9]							
...								
21	$\Delta = 3$ [22]							

Table 1: The latest results with a coefficient 1 before Δ in the upper bound of χ^2.

For example, the result from line "7" and column "$\Delta + 1$" from Table 1 reads as follows : "*every planar graph G of girth at least 7 and of Δ at least 16 satisfies* $\chi^2(G) \leq \Delta + 1$". The crossed out cases in the first column correspond to the fact that, for $g_0 \leq 6$, there are planar graphs G with $\chi^2(G) = \Delta + 2$ for arbitrarily large Δ [4, 15]. There exists a construction for $\Delta = 3$, girth 11, and $\chi^2 \geq \Delta + 2$ detailed in [22]. In Figure 4(i), we show a simple planar graph with girth 4 and $\chi^2 \geq \Delta + 3$ for all $\Delta \geq 2$, which justifies the crossed out cases in the second column. In Figure 4(ii), we show a graph with $\Delta = 4$, girth 5, and $\chi^2 \geq 7$. For the first square of the third column, Figure 3 shows graphs with $\chi^2 \geq \Delta + 4$ for $3 \leq \Delta \leq 7$ and starting from $\Delta \geq 8$, the graph in Figure 2(i) verifies $\chi^2 \geq \Delta + 5$. Similarly, the rest of the crossed out values of Δ comes from the constructions in Figure 2 and Figure 3.

We are interested in the case $\chi^2(G) \leq \Delta + 2$. More particularly, for a fixed Δ_0, we want to find the lowest value g_0 such that planar graphs G with maximum degree Δ_0 and girth at least g_0 verify $\chi^2(G) \leq \Delta_0 + 2$.

In what follows, we concentrate on the case $\Delta_0 = 4$. In Figure 4, we provide some simple graphs that give us a lower bound on g_0 depending on Δ_0.

For Figure 4(i), u, v and their common neighbors must all be colored differently, say they are colored 1 to $\Delta + 1$. The two remaining vertices must be colored with two different colors. As each of them sees every color from 1 to $\Delta + 1$, they must be colored with $\Delta + 2$ and $\Delta + 3$.

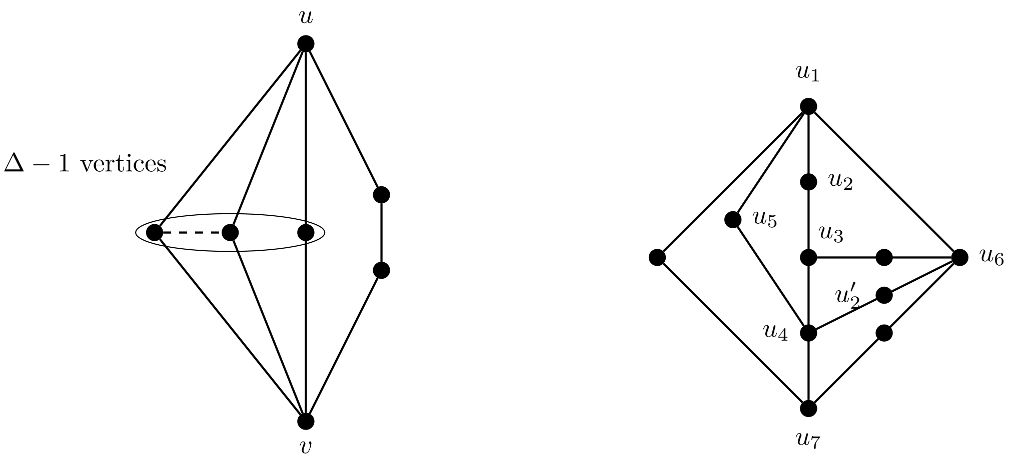
For Figure 4(ii), suppose that it is 2-distance colorable with only six colors. Vertices u_1, u_2, u_3, u_4, and u_5 form a cycle of length 5 so they must all be colored differently, say u_i is colored i for $1 \leq i \leq 5$. Vertex u_6 sees every color from 1 to 5 so it must be colored 6. Vertex u_2' sees every color except 2 so it must be colored 2. Finally, u_7 sees every color from 1 to 6, which is a contradiction.

Now, for the upper bound on g_0, we are going to prove that $g_0 \leq 10$. In other words,

Theorem 3. *If G is a planar graph with $g(G) \geq 10$ and $\Delta(G) \geq 4$, then $\chi^2(G) \leq \Delta(G) + 2$.*

As the following results are already known:

[1] Figure 3
[2] Figure 2(i)
[3] Figure 2(ii)
[4] Figure 4(ii)
[5] Corollaries of more general colorings of planar graphs.
[6] Corollaries of 2-distance list-colorings of planar graphs.
[7] Corollaries of 2-distance list-colorings of graphs with a bounded maximum average degree.
[8] Corollary of our result.
[9] Corollaries of 2-distance colorings of graphs with a bounded maximum average degree.



(i) A graph with girth 4 and $\chi^2 \geq \Delta + 3$.

(ii) A graph with $\Delta = 4$, girth 5, and $\chi^2 \geq 7$.

Figure 4: Graphs with $\chi^2 \geq \Delta + 3$.

Theorem 4 (Bu *et al.* [6]). *If G is a planar graph with $g(G) \geq 9$ and $\Delta(G) = 5$, then $\chi^2(G) \leq \Delta(G) + 2$.*

Theorem 5 (La and Montassier [24]). *If G is a planar graph with $g(G) \geq 8$ and $\Delta(G) \geq 6$, then $\chi^2(G) \leq \Delta(G) + 2$.*

we only need to prove that:

Theorem 6. *If G is a planar graph with $g(G) \geq 10$ and $\Delta(G) = 4$, then $\chi^2(G) \leq 6$.*

We will be proving a slightly stronger version which states:

Theorem 7. *If G is a graph with $\text{mad}(G) < \frac{5}{2}$, $g(G) \geq 10$ and $\Delta(G) = 4$, then $\chi_l^2(G) \leq 6$.*

The latter also improves upon a result from [10].

Theorem 8 (Cranston *et al.* [10]). *If G is a planar graph with $g(G) \geq 11$ and $\Delta(G) = 4$, then $\chi_l^2(G) \leq 6$.*

In Section 2, we present the proof of Theorem 7 using the well-known discharging method.

2 Proof of Theorem 7

Notations and drawing conventions. For $v \in V(G)$, the *2-distance neighborhood* of v, denoted $N_G^*(v)$, is the set of 2-distance neighbors of v, which are vertices at distance at most two from v, not including v. We also denote $d_G^*(v) = |N_G^*(v)|$. We will drop the subscript and the argument when it is clear from the context. Also for conciseness, from now on, when we say "to color" a vertex, it means to color such vertex differently from all of its colored neighbors at distance at most two. Similarly, any considered coloring will be a 2-distance list-coloring. We will also say that a vertex u "sees" another vertex v if u and v are at distance at most 2 from each other.

Some more notations:

- A *d-vertex* (d^+-vertex, d^--vertex) is a vertex of degree d (at least d, at most d). A *$(d \leftrightarrow e)$-vertex* is a vertex of degree between d and e included.
- A *k-path* (k^+-path, k^--path) is a path of length $k+1$ (at least $k+1$, at most $k+1$) where the k internal vertices are 2-vertices and the endvertices are 3^+-vertices.
- A *$(k_1, k_2, \ldots, k_d)$-vertex* is a d-vertex incident to d different paths, where the i^{th} path is a k_i-path for all $1 \leq i \leq d$.

As a drawing convention for the rest of the figures, black vertices will have a fixed degree, which is represented, and white vertices may have a higher degree than what is drawn.

Let G be a counterexample to Theorem 7 with the fewest number of vertices. The purpose of the proof is to prove that G cannot exist. In the following sections, we will study the structural properties of G (Section 2.2), then, we will apply a discharging procedure (Section 2.3).

2.1 Useful observations

Before studying the structural properties of G, we will introduce some useful observations and lemmas that will be the core of the reducibility proofs of our configurations.

For a vertex u, let $L(u)$ denote the set of available colors for u. For convenience, the lower bound on $|L(u)|$ will be depicted on the figures below the corresponding vertex u.

Lemma 9. *Every graph with list assignment L depicted in Figure 5 is L-list-colorable.*

Proof. In the following proofs, whenever the size of a list $|L(u)| \geq i$, we assume that $|L(u)| = i$ by removing the extra colors from the list.

(i) If $L(u_1) = L(u_2)$, then we color u_3 with a color in $L(u_3) \setminus L(u_2)$, followed by u_4, u_2, and u_1 in this order. If $L(u_1) \neq L(u_2)$, then we color u_2 with a color in $L(u_2) \setminus L(u_1)$, followed by u_4, u_3, and u_1 in this order.

(ii) First, we claim the following:

- $L(u_5) \cap L(u_1) = \emptyset$ and $L(u_5) \cap L(u_2) = \emptyset$. Suppose by contradiction that there exists $x \in L(u_5) \cap L(u_1)$, we color u_1 and u_5 with x, then u_2, u_3, u_3'', u_3', and u_4 in this order. The same argument holds for $L(u_5) \cap L(u_2)$.
- $L(u_5) \cap L(u_3') = \emptyset$. Suppose by contradiction that there exists $x \in L(u_5) \cap L(u_3')$, we color u_3' and u_5 with x. Observe that $L(u_5) \cap L(u_1) = \emptyset$ and $L(u_5) \cap L(u_2) = \emptyset$. So, we color u_3'', u_3, u_2, u_1, and u_4 in this order.
- $L(u_5) \cap L(u_3) = \emptyset$. Otherwise, we color u_3 with $x \in L(u_5) \cap L(u_3)$. Observe that $L(u_5) \cap L(u_1) = \emptyset$, $L(u_5) \cap L(u_2) = \emptyset$, and $L(u_5) \cap L(u_3') = \emptyset$. So, we color u_5, u_3'', u_2, u_1, u_4, and u_3' in this order.

Since $L(u_5) \cap L(u_3) = \emptyset$, we color u_2, u_1, u_3, u_3'', u_3', u_4, and u_5 in this order.

(iii) First, we claim that $L(v_3') \subset L(v_3)$. Otherwise, we color v_3' with $x \in L(v_3') \setminus L(v_3)$. Then, we color everything else except v_3 thanks to Figure 5ii. We finish by coloring v_3.

Now, we color u_3'' with $x \in L(u_3'') \setminus L(v_3)$. Since $L(v_3') \subset L(v_3)$, $x \notin L(v_3')$. Thus, we color u_2, u_1, u_3, u_5, u_4, u_3', v_3', and v_3 in this order.

(iv) First, we claim that $L(v_3') \subset L(v_3)$ and $L(v_5') \subset L(v_5)$. Suppose by contradiction that there exists $x \in L(v_5') \setminus L(v_5)$. We color v_5' with x. Then, we color everything else except v_5 thanks to Figure 5iii. We finish by coloring v_5. Symmetrically, the same holds for $L(v_3') \subset L(v_3)$.

Now, we color u_3'' with $x \in L(u_3'') \setminus L(v_3)$. Since $L(v_3') \subset L(v_3)$, $x \notin L(v_3')$. Similarly, we color u_5 with $y \in L(u_5) \setminus L(v_5)$. We finish by coloring u_3, u_2, u_1, u_3', v_3', v_3, u_4, v_5', and v_5 in this order.

(v) First, we claim the following:

- $L(u_2) \cap L(u_5) = \emptyset$ and $L(u_2) \cap L(u_3'') = \emptyset$. Suppose by contradiction that there exists $x \in L(u_2) \cap L(u_5)$. We color u_2 and u_5 with x, then u_1, u_3, u_3'', u_3', and v_4 in this order. Symmetrically, the same holds for $L(u_2) \cap L(u_3'')$.
- $L(u_3) \subset (L(u_5) \cup L(u_3''))$. Suppose by contradiction that there exists $x \in L(u_3) \setminus (L(u_5) \cup L(u_3''))$. We color u_3 with x, then u_1, u_2, u_3', u_3'', u_4, then u_5 in this order.
- $L(u_3') \cap L(u_5) = \emptyset$ and $L(u_4) \cap L(u_3'') = \emptyset$. Suppose by contradiction that there exists $x \in L(u_3') \cap L(u_5)$. We color u_3' and u_5 with x. Observe that $x \notin L(u_2)$ since $L(u_2) \cap L(u_5) = \emptyset$. Thus, we color u_3'', u_3, u_1, u_2 and u_4 in this order. Symmetrically, the same holds for $L(u_4) \cap L(u_3'')$.

Since $L(u_3) \subset (L(u_5) \cup L(u_3''))$, $|L(u_3)| = 3$, and $|L(u_5)| = |L(u_3'')| = 2$, there must exist $x \in L(u_5) \cap L(u_3)$. In addition, $x \notin L(u_3')$ as $L(u_3') \cap L(u_5) = \emptyset$. Thus, we color u_3 with x, then u_1, u_2, u_5, u_4, u_3'', and u_3' in this order.

(vi) First, we claim that $L(v_3') \subset L(v_3)$. Otherwise, we color v_3' with $x \in L(v_3') \setminus L(v_3)$. Then, we color everything else except v_3 thanks to Figure 5v. We finish by coloring v_3.

Now, we color u_3'' with $x \in L(u_3'') \setminus L(v_3)$. Since $L(v_3') \subset L(v_3)$, $x \notin L(v_3')$. Thus, we color u_1, u_3, u_2, u_5, u_4, u_3', v_3', and v_3 in this order.

(vii) First, we claim that $L(v_3') \subset L(v_3)$ and $L(v_5') \subset L(v_5)$. Suppose by contradiction that there exists $x \in L(v_5') \setminus L(v_5)$. We color v_5' with x. Then, we color everything else except v_5 thanks to Figure 5vi. We finish by coloring v_5. Symmetrically, the same holds for $L(v_3') \subset L(v_3)$.

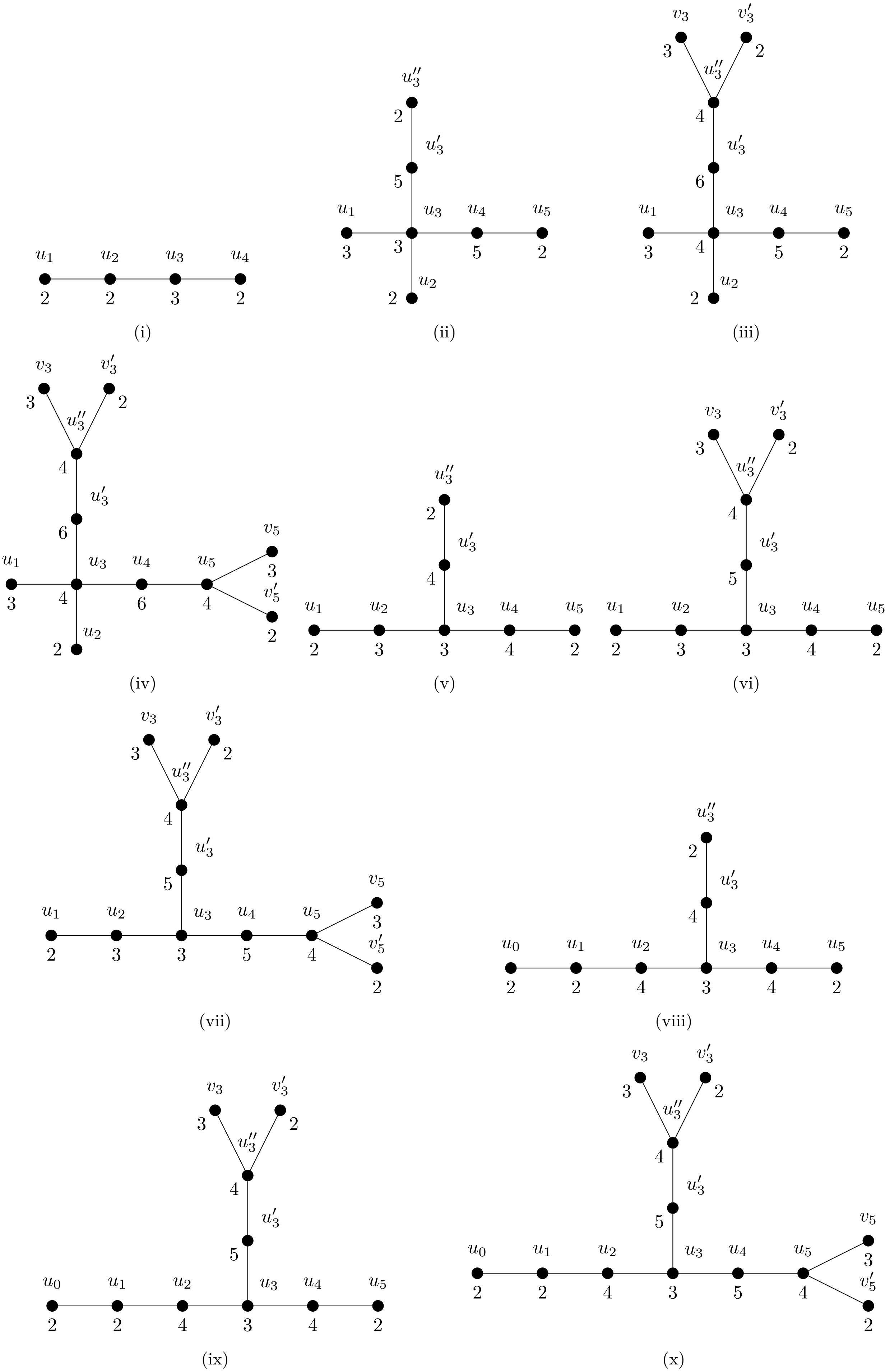


Figure 5: List-colorable graphs.

Now, we color u_3'' with $x \in L(u_3'') \setminus L(v_3)$. Since $L(v_3') \subset L(v_3)$, $x \notin L(v_3')$. Similarly, we color u_5 with $y \in L(u_5) \setminus L(v_5)$. We finish by coloring u_3, u_1, u_2, u_3', v_3', v_3, u_4, v_5', and v_5 in this order.

(viii) First, we claim that $L(u_0) = L(u_1)$. Otherwise, we color u_0 with $x \in L(u_0) \setminus L(u_1)$. Then, we color the rest thanks to Figure 5v.

Since $L(u_0) = L(u_1)$, we can restrict $L(u_2)$ to $L'(u_2) = L(u_2) \setminus L(u_1)$ and observe that, if we can color everything (where u_2 has list $L'(u_2)$) except u_0 and u_1, then we can always finish by coloring u_1 and u_0 in this order.

Let us show that everything except u_0 and u_1 can be colored first with the new list $L'(u_2)$ for u_2. We claim the following:

- $L(u_5) \cap L'(u_2) = \emptyset$. Otherwise, we color u_2 and u_5 with $x \in L(u_5) \cap L'(u_2)$. Then, we color u_3, u_3'', u_3', and u_4 in this order.
- $L(u_5) \cap L(u_3') = \emptyset$. Otherwise, we color u_3' and u_5 with $x \in L(u_5) \cap L(u_3')$. Observe that $L(u_5) \cap L'(u_2) = \emptyset$ so $x \notin L'(u_2)$. So, we color u_3'', u_3, u_2, and u_4 in this order.
- $L(u_5) \cap L(u_3) = \emptyset$. Otherwise, we color u_3 with $x \in L(u_5) \cap L(u_3)$. Observe that $L(u_5) \cap L'(u_2) = \emptyset$ and $L(u_5) \cap L(u_3') = \emptyset$ so $x \notin L'(u_2) \cup L(u_3')$. So, we color u_5, u_3'', u_2, u_4, and u_3' in this order.

Since $L(u_5) \cap L(u_3) = \emptyset$, we color u_2, u_3, u_3'', u_3', u_4, and u_5 in this order.

(ix) First, we claim that $L(v_3') \subset L(v_3)$. Otherwise, we color v_3' with $x \in L(v_3') \setminus L(v_3)$. Then, we color eveything else except v_3 thanks to Figure 5viii. We finish by coloring v_3.

Now, we color u_3'' with $x \in L(u_3'') \setminus L(v_3)$. Since $L(v_3') \subset L(v_3)$, $x \notin L(v_3')$. Thus, we color u_0, u_1, u_3, u_2, u_5, u_4, u_3', v_3', and v_3 in this order.

(x) First, we claim that $L(v_3') \subset L(v_3)$ and $L(v_5') \subset L(v_5)$. Suppose by contradiction that there exists $x \in L(v_5') \setminus L(v_5)$. We color v_5' with x. Then, we color everything else except v_5 thanks to Figure 5ix. We finish by coloring v_5. Symmetrically, the same holds for $L(v_3') \subset L(v_3)$.

Now, we color u_3'' with $x \in L(u_3'') \setminus L(v_3)$. Since $L(v_3') \subset L(v_3)$, $x \notin L(v_3')$. Similarly, we color u_5 with $y \in L(u_5) \setminus (L(v_5) \cup L(v_5'))$. We finish by coloring u_3, u_1, u_0, u_2, u_3', v_3', v_3, u_4, v_5', and v_5 in this order.

□

2.2 Structural properties of G

Lemma 10. *Graph G is connected.*

Proof. Otherwise a component of G would be a smaller counterexample. □

Lemma 11. *The minimum degree of G is at least 2.*

Proof. By Lemma 10, the minimum degree is at least 1 or G would be a single isolated vertex contradicting $\Delta(G) = 4$. If G contains a degree 1 vertex v, then we can simply remove v and 2-distance color the resulting graph, which is possible by minimality of G. Then, we add v back and extend the coloring (at most 4 constraints and 6 colors). □

(i) A 3^+-path. (ii) A 2-path incident to a 3-vertex.

Figure 6: Path cases.

Lemma 12. *Graph G does not contain any 3^+-path.*

Proof. Suppose by contradiction that G does contain three consecutives 2-vertices uvw (see Figure 6i). It suffices to color $G - \{u, v, w\}$ by minimality of G, then we can extend the coloring to the remaining vertices by coloring u, w, then v in this order. This is possible since u, v, and w have respectively at least 2, 4, and 2 colors left available. □

Lemma 13. *An endvertex of a* 2*-path must be a* 4*-vertex.*

Proof. Suppose by contradiction that there exists a 2-path $uvwx$ where $d(u) = 3$ (see Figure 6ii). We color $G - \{v, w\}$ by minimality of G. Then, we color w and v in this order since they have respectively at least 1 and 2 colors left available. $\square$

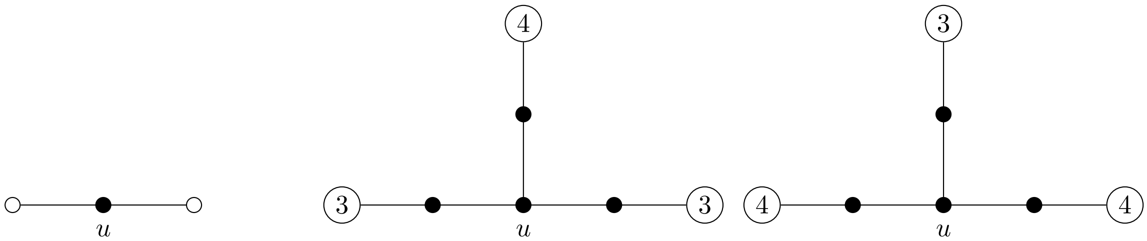
Let us define some nomenclatures.

Definition 14. *Let u be a* $(1,1,1)$*-vertex and let v, w, and x be the other endvertices of the* 1*-paths incident to u. We call u*

- *a* small $(1,1,1)$*-vertex, if v, w, and x are all* 3*-vertices.*
- *a* medium $(1,1,1)$*-vertex, if exactly one of v, w, and x is a* 4*-vertex.*
- *a* large $(1,1,1)$*-vertex, if exactly two of v, w, and x are* 4*-vertices.*
- *a* huge $(1,1,1)$*-vertex, if v, w, and x are all* 4*-vertices.*

Definition 15. *Let u be a* $(1,1,0)$*-vertex with a* 3*-neighbor and let u share a common* 2*-neighbor with a small* $(1,1,1)$*-vertex. We call u a* special $(1,1,0)$-vertex.

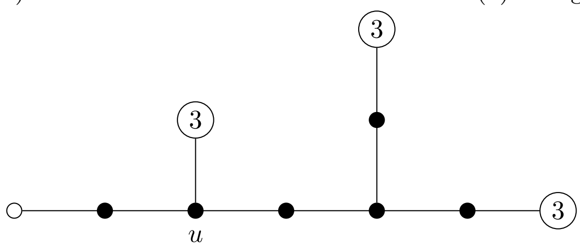
Definition 16. *We call u a* light *vertex if u is a 2-vertex, a medium* $(1,1,1)$*-vertex, or a large* $(1,1,1)$*-vertex.*



(i) A 2-vertex. (ii) A medium $(1,1,1)$-vertex. (iii) A large $(1,1,1)$-vertex.

Figure 7: Light vertices.

(i) A small $(1,1,1)$-vertex. (ii) A huge $(1,1,1)$-vertex.



(iii) A special $(1,1,0)$-vertex.

Figure 8

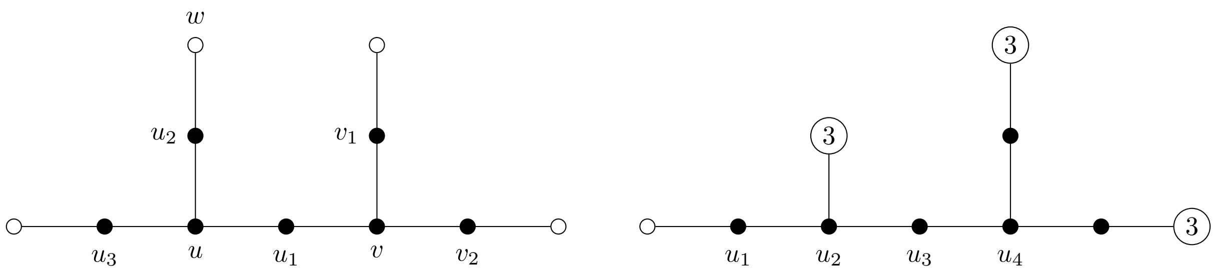
Lemma 17. *If two* $(1,1,1)$*-vertices share a common* 2*-neighbor, then they must be large* $(1,1,1)$*-vertices.*

Proof. Let uu_1v be a 1-path where both u and v are $(1,1,1)$-vertices. Let u_2 and u_3 be u's other 2-neighbors and v_1 and v_2 be v's other 2-neighbors. Let w be the other endvertex of wu_2u. Since $g(G) \geq 10$, all named vertices are distinct. See Figure 9i.

Suppose by contradiction that $d(w) = 3$, in other words, that u is not a large $(1,1,1)$-vertex. We color $G - \{u, u_1, u_2, u_3\}$ by minimality of G and uncolor v. Then, we color u_3 and finish with u_2, u, u_1, and v thanks to Figure 5i. □

Lemma 18. *A special $(1,1,0)$-vertex must share a 2-neighbor with a 4-vertex.*

Proof. Let $u_2u_3u_4$ be a 1-path where u_2 is a special $(1,1,0)$-vertex and u_4 is a small $(1,1,1)$-vertex. Let $u_1 \neq u_3$ be u_2's other 2-neighbor. See Figure 9ii. Suppose by contradiction that u_1 is adjacent to another 3-vertex. Since $g(G) \geq 10$, all vertices that see u_3 are distinct. We color $G - \{u_1, u_2, u_3\}$ by minimality of G and we uncolor u_4. We extend the coloring to u_1, u_2, u_3, and u_4 thanks to Figure 5i. □



(i) Large 3-vertices case. (ii) Special $(1,1,0)$-vertex case.

Figure 9: 3-vertices case.

Lemma 19. *Let u be incident to four 1^+-paths uu_iv_i for $1 \leq i \leq 4$. If v_1 and v_2 are light vertices, then $d(v_3) \geq 4$ and $d(v_4) \geq 4$.*

Proof. Suppose by contradiction that v_1 and v_2 are light vertices but $d(v_3) \leq 3$. (see Figure 10i). The proof will proceed as follows. For each combination of light vertices v_1 and v_2, we will define H a subgraph of G. We color $G - H$ by minimality of G. Then, let $L(x)$ be the list of remaining colors for every $x \in V(H)$. We will use Figure 5 to show that H is always colorable, thus obtaining a valid coloring G, which is a contradiction. Observe that $g(G) \geq 10$ so $g(H) \geq 10$, which means that, in the following subgraphs, every considered vertex will be distinct and their neighborhood at distance at most 2 will be represented exactly by the subgraphs in Figure 5.

- If v_1 and v_2 are 2-vertices, then $H = \{u, u_1, u_2, u_3, u_4, v_1, v_2\}$ is colorable thanks to Figure 5ii.
- If v_1 is a 2-vertex and v_2 is a medium or large $(1,1,1)$-vertex, then $H = \{u, u_1, u_2, u_3, u_4, v_1, v_2\} \cup N_G(v_2)$ is colorable thanks to Figure 5iii.
- If v_1 and v_2 are medium or large $(1,1,1)$-vertices, then $H = \{u, u_1, u_2, u_3, u_4, v_1, v_2\} \cup N_G(v_1) \cup N_G(v_2)$ is colorable thanks to Figure 5iv.

By symmetry, the same holds for $d(v_4)$. □

Lemma 20. *Let u be a 4-vertex with a 3-neighbor and let u be incident to three 1^+-paths uu_iv_i for $1 \leq i \leq 3$. If v_1 and v_2 are light vertices, then v_3 is a non-special $(1,1,0)$-vertex, a $(1,0,0)$-vertex, or a 4-vertex.*

Proof. Suppose by contradiction that v_1 and v_2 are light vertices but v_3 is a 2-vertex, a special $(1,1,0)$-vertex, or a $(1,1,1)$-vertex (see Figure 10ii).

We will proceed like the proof of Lemma 19 by defining a certain subgraph H, coloring $G - H$ and extending it to H by using Figure 5.

Let v_3 be a special $(1,1,0)$-vertex or a light vertex. When v_3 is a 3-vertex, note that v_3 always has a 2-neighbor v that is adjacent to another 3-vertex different from v_3. We define $N'_G(v_3) = \emptyset$ when v_3 is a 2-vertex and $N'_G(v_3) = \{v\}$ when v_3 is a 3-vertex.

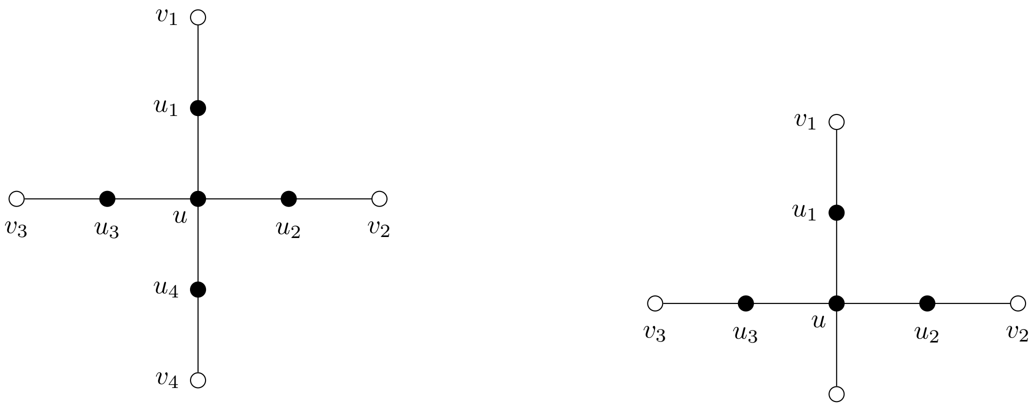
- If v_1 and v_2 are 2-vertices, then $H = \{u, u_1, u_2, u_3, v_1, v_2, v_3\} \cup N'_G(v_3)$ is colorable thanks to Figure 5viii when $N'_G(v_3) = \{v\}$ or Figure 5v otherwise.
- If v_1 is a 2-vertex and v_2 is a medium or large $(1,1,1)$-vertex, then $H = \{u, u_1, u_2, u_3, v_1, v_2, v_3\} \cup N'_G(v_3) \cup N_G(v_2)$ is colorable thanks to Figure 5ix when $N'_G(v_3) = \{v\}$ or Figure 5vi otherwise.

- If v_1 and v_2 are medium or large $(1,1,1)$-vertices, then $H = \{u, u_1, u_2, u_3, v_1, v_2, v_3\} \cup N'_G(v_3) \cup N_G(v_1) \cup N_G(v_2)$ is colorable thanks to Figure 5x when $N'_G(v_3) = \{v\}$ or Figure 5vii otherwise.

Let v_3 be a huge $(1,1,1)$-vertex. In the following cases, we actually remove everything in H except v_3, color $G - (H \setminus \{v_3\})$, then uncolor v_3, and we extend the coloring to H.

- If v_1 and v_2 are 2-vertices, then $H = \{u, u_1, u_2, u_3, v_1, v_2, v_3\}$ is colorable thanks to Figure 5v.
- If v_1 is a 2-vertex and v_2 is a medium or large $(1,1,1)$-vertex, then $H = \{u, u_1, u_2, u_3, v_1, v_2, v_3\} \cup N_G(v_2)$ is colorable thanks to Figure 5vi.
- If v_1 and v_2 are medium or large $(1,1,1)$-vertices, then $H = \{u, u_1, u_2, u_3, v_1, v_2, v_3\} \cup N_G(v_1) \cup N_G(v_2)$ is colorable thanks to Figure 5vii.

□



(i) $(1,1,1,1)$-vertex case.

(ii) $(1,1,1,0)$-vertex case.

Figure 10: 4-vertices case.

2.3 Discharging rules

Since $\text{mad}(G) < \frac{5}{2}$, we must have

$$\sum_{u \in V(G)} (4d(u) - 10) < 0 \tag{1}$$

We assign to each vertex u the charge $\mu(u) = 4d(u) - 10$. To prove the non-existence of G, we will redistribute the charges preserving their sum and obtaining a non-negative total charge, which will contradict Equation (1).

We then apply the following discharging rules:

R0 Every 3^+-vertex gives 1 to each 2-vertex on its incident 1^+-paths.

R1 Every 4-vertex gives 1 to each of its 3-neighbors.

R2 Let vtu be a 1-path.

(i) If v is a $(1,1^-,0)$-vertex and u is a small $(1,1,1)$-vertex, then v gives $\frac{1}{3}$ to u.

(ii) If v is a 4-vertex and u is a medium $(1,1,1)$-vertex, then v gives 1 to u.

(iii) If v is a 4-vertex and u is a large $(1,1,1)$-vertex, then v gives $\frac{1}{2}$ to u.

(iv) If v is a 4-vertex and u is a huge $(1,1,1)$-vertex, then v gives $\frac{1}{3}$ to u.

(v) If v is a 4-vertex and u is a special $(1,1,0)$-vertex, then v gives $\frac{1}{3}$ to u.

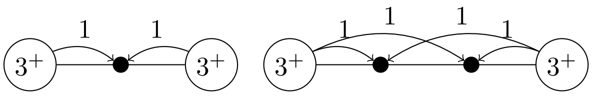


Figure 11: **R0**.

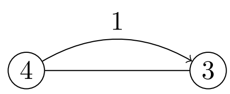
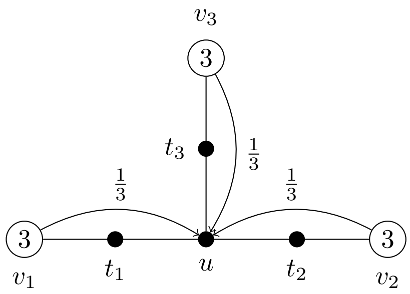
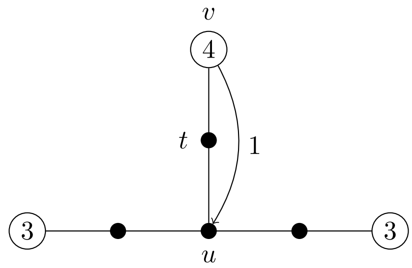


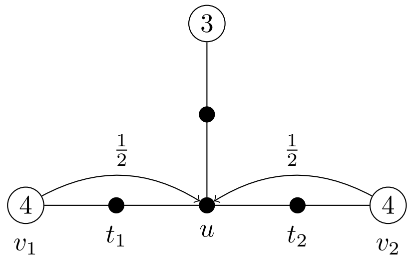
Figure 12: **R1**.



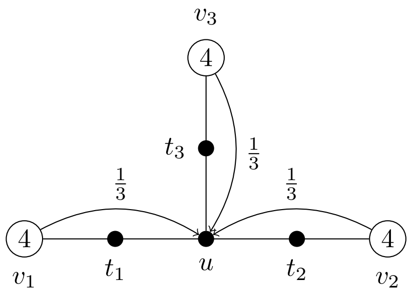
(i) Small $(1,1,1)$-vertex case:
for $1 \leq i \leq 3$, $v_i \neq (1,1,1)$-vertex.



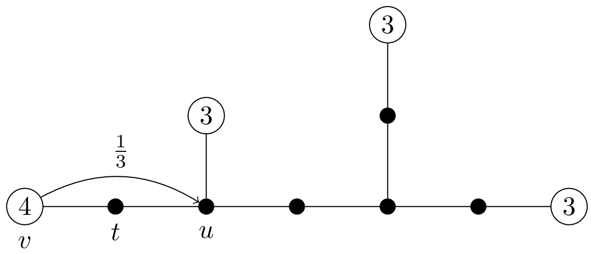
(ii) Medium $(1,1,1)$-vertex case.



(iii) Large $(1,1,1)$-vertex case.



(iv) Huge $(1,1,1)$-vertex case.



(v) Special $(1,1,0)$-vertex case.

Figure 13: **R2**.

2.4 Verifying that charges on each vertex are non-negative

Let μ^* be the assigned charges after the discharging procedure. In what follows, we will prove that:

$$\forall u \in V(G), \mu^*(v) \geq 0.$$

Let u be a vertex in $V(G)$.
Case 1: If $d(u) = 2$, then u receives charge 1 from each endvertex of the path it lies on by **R0**. Thus, we get:

$$\mu^*(u) = \mu(u) + 2 \cdot 1 = 4 \cdot 2 - 10 + 2 = 0.$$

Case 2: If $d(u) = 3$, then recall that $\mu(u) = 4 \cdot 3 - 10 = 2$. Moreover, u cannot be incident to any 2^+-paths due to Lemmas 12 and 13. Now, we distinguish the following cases:

- If u is a $(1,1,1)$-vertex, then u only gives charge to its 2-neighbors, more precisely 1 to each of its 2-neighbors by **R0**. Let v, w, and x be the other endvertices of the 1-paths incident to u.

  If u is a small $(1,1,1)$-vertex, then observe that v, w, and x are all $(1,1^-,0)$-vertices as they cannot be $(1,1,1)$-vertices due to Lemma 17. As a result, u receives $\frac{1}{3}$ from each of v, w, and x by **R2**(i). Hence,

  $$\mu^*(u) = 2 - 3 \cdot 1 + 3 \cdot \frac{1}{3} = 0.$$

  If u is a medium $(1,1,1)$-vertex, then u receives 1 from one of v, w, and x by **R2**(ii). Hence,

  $$\mu^*(u) = 2 - 3 \cdot 1 + 1 = 0.$$

  If u is a large $(1,1,1)$-vertex, then u receives $\frac{1}{2}$ twice from v, w, and x by **R2**(iii). Hence,

  $$\mu^*(u) = 2 - 3 \cdot 1 + 2 \cdot \frac{1}{2} = 0.$$

  If u is a huge $(1,1,1)$-vertex, then u receives $\frac{1}{3}$ from each of v, w, and x by **R2**(iv). Hence,

  $$\mu^*(u) = 2 - 3 \cdot 1 + 3 \cdot \frac{1}{3} = 0.$$

- If u is a $(1,1,0)$-vertex, then u gives 1 to each of its 2-neighbors by **R0**. Let t be its 3^+-neighbor and let v and w be the other endvertices of the 1-paths incident to u. First, observe that if neither v nor w is a small $(1,1,1)$-vertex, then we would have

$$\mu^*(u) \geq 2 - 2 \cdot 1 = 0.$$

  So, say v is a small $(1,1,1)$-vertex, in which case, u gives $\frac{1}{3}$ to v by **R2**(i).

  If $d(t) = 4$, then u receives 1 from t by **R1**. Moreover, at worst, u also gives $\frac{1}{3}$ to w by **R2**(i). To sum up,

$$\mu^*(u) \geq 2 - 2 \cdot 1 + 1 - 2 \cdot \frac{1}{3} = \frac{1}{3}.$$

  If $d(t) = 3$, then w must be a 4-vertex by Lemma 18. In other words, u is a special $(1,1,0)$-vertex. Thus, u receives $\frac{1}{3}$ from w by **R3**. To sum up,

$$\mu^*(u) \geq 2 - 2 \cdot 1 - \frac{1}{3} + \frac{1}{3} = 0.$$

- If u is a $(1,0,0)$-vertex, then at worst, u gives 1 to its 2-neighbor by **R0** and $\frac{1}{3}$ to the other endvertex of its incident 1-path by **R2**(i). Thus,

$$\mu^*(u) \geq 2 - 1 - \frac{1}{3} = \frac{2}{3}.$$

- If u is a $(0,0,0)$-vertex, then none of the discharging rules apply. Thus,

$$\mu^*(u) = \mu(u) = 2.$$

Case 3: If $d(u) = 4$, then recall that $\mu(u) = 4 \cdot 4 - 10 = 6$. Observe that u gives away at most 2 per incident 0^+-path. Indeed, there are no 3^+-paths by Lemma 12. More precisely, u gives:

2 to a 2-path by **R0**.

2 to a 1-path with a medium $(1,1,1)$-endvertex by **R0** and **R2**(ii).

$\frac{3}{2}$ to a 1-path with a large $(1,1,1)$-endvertex by **R0** and **R2**(iii).

$\frac{4}{3}$ to a 1-path with a huge $(1,1,1)$-endvertex by **R0** and **R2**(iv).

$\frac{4}{3}$ to a 1-path with a special $(1,1,0)$-endvertex by **R0** and **R2**(v).

1 to a 1-path with a non-special $(1,1,0)$-endvertex, a $(1,0,0)$-endvertex or a 4-endvertex by **R0**.

1 to a 3-neighbor by **R1**.

Observe that u only gives more than $\frac{4}{3}$ to an incident path when the neighbor at distance 2 on that path is a light vertex (Definition 16). Now, we distinguish the following cases:

- If u is a $(1^+,1^+,1^+,1^+)$-vertex, then let uu_iv_i be the 1^+-paths incident to u for $1 \leq i \leq 4$.

  If at most one of the v_i's is a light vertex, then at worst

$$\mu^*(u) \geq 6 - 2 - 3 \cdot \frac{4}{3} = 0.$$

  If at least two of the v_i's are light vertices, say v_1 and v_2, then $d(v_3) \geq 4$ and $d(v_4) \geq 4$ due to Lemma 19. As a result, u gives only 1 to each of uu_3v_3 and uu_4v_4. Thus, at worst we get

$$\mu^*(u) \geq 6 - 2 \cdot 2 - 2 \cdot 1 = 0.$$

- If u is a $(1^+,1^+,1^+,0)$-vertex with a 4-neighbor, then u does not give anything to its 4-neighbor. Thus, at worst we have

$$\mu^*(u) \geq 6 - 3 \cdot 2 = 0.$$

- If u is a $(1^+, 1^+, 1^+, 0)$-vertex with a 3-neighbor, then let uu_iv_i be the 1^+-paths incident to u for $1 \leq i \leq 3$. We know that u always give 1 to its 3-neighbor.

  If at most one of the v_i's is a light vertex, then at worst

$$\mu^*(u) \geq 6 - 1 - 2 - 2 \cdot \frac{4}{3} = \frac{1}{3}.$$

  If at least two of the v_i's are light vertices, say v_1 and v_2, then v_3 must be a non-special $(1, 1, 0)$-endvertex, a $(1, 0, 0)$-endvertex or a 4-endvertex due to Lemma 20. As a result, u gives only 1 to uu_3v_3. Thus, at worst we get

$$\mu^*(u) \geq 6 - 1 - 2 \cdot 2 - 1 = 0.$$

- If u is a $(0^+, 0^+, 0, 0)$-vertex, then at worst we have

$$\mu^*(u) \geq 6 - 2 \cdot 2 - 2 \cdot 1 = 0.$$

To conclude, we started with a charge assignment with a negative total sum, but after the discharging procedure, which preserved that sum, we end up with a non-negative one, which is a contradiction. In other words, there exists no counter-example to Theorem 7.

Remarks

Graphs of with $mad < \frac{5}{2}$ contains all planar graphs with girth at least 10. The condition on the girth in Theorem 7 only serves to simplify the proof of the reducibility of certain configurations as it guarantees that some vertices must be distinct. Here, we chose girth 10 as it would coincide with the girth of the subclass of planar graphs, but girth 9 suffices to guarantee the desired property. One can also strengthen Theorem 7 by removing this condition and study the cases where some vertices in our configurations coincide.

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