2-Distance List (Δ+3) -Coloring of Sparse Graphs

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2-distance $(\Delta + 2)$-coloring of sparse graphs

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Abstract

A 2-distance k-coloring of a graph is a proper k-coloring of the vertices where vertices at distance at most 2 cannot share the same color. We prove the existence of a 2-distance $(\Delta+2)$-coloring for graphs with maximum average degree less than $\frac{8}{3}$ (resp. $\frac{14}{5}$) and maximum degree $\Delta \geq 6$ (resp. $\Delta \geq 10$). As a corollary, every planar graph with girth at least 8 (resp. 7) and maximum degree $\Delta \geq 6$ (resp. $\Delta \geq 10$) admits a 2-distance $(\Delta + 2)$-coloring.

1 Introduction

A k-*coloring* of the vertices of a graph $G = (V, E)$ is a map $\phi : V \to \{1, 2, \dots, k\}$. A k-coloring ϕ is a *proper coloring*, if and only if, for all edge $xy \in E, \phi(x) \neq \phi(y)$. In other words, no two adjacent vertices share the same color. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k such that G has a proper k-coloring. A generalization of k-coloring is k-list-coloring. A graph G is L-*list colorable* if for a given list assignment $L = \{L(v) : v \in V(G)\}$ there is a proper coloring ϕ of G such that for all $v \in V(G), \phi(v) \in L(v)$. If G is L-list colorable for every list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$, then G is said to be k-*choosable* or k-*list-colorable*. The *list chromatic number* of a graph G is the smallest integer k such that G is k-choosable. List coloring can be very different from usual coloring as there exist graphs with a small chromatic number and an arbitrarily large list chromatic number.

In 1969, Kramer and Kramer introduced the notion of 2-distance coloring [19, 20]. This notion generalizes the "proper" constraint (that does not allow two adjacent vertices to have the same color) in the following way: a 2-*distance* k-*coloring* is such that no pair of vertices at distance at most 2 have the same color. The 2-*distance chromatic number* of G, denoted by $\chi^2(G)$, is the smallest integer k such that G has a 2-distance k-coloring. Similarly to proper k-list-coloring, one can also define 2-*distance* k-*list-coloring* and a 2-*distance list chromatic number*.

For all $v \in V$, we denote $d_G(v)$ the degree of v in G and by $\Delta(G) = \max_{v \in V} d_G(v)$ the maximum degree of a graph G. For brevity, when it is clear from the context, we will use Δ (resp. $d(v)$) instead of $\Delta(G)$ (resp. $d_G(v)$). One can observe that, for any graph G, $\Delta + 1 \leq \chi^2(G) \leq \Delta^2 + 1$. The lower bound is trivial since, in a 2-distance coloring, every neighbor of a vertex v with degree Δ, and v itself must have a different color. As for the upper bound, a greedy algorithm shows that $\chi^2(G) \leq \Delta^2 + 1$. Moreover, that upper bound is tight for some graphs, for example, Moore graphs of type $(\Delta, 2)$, which are graphs where all vertices have degree Δ, are at distance at most two from each other, and the total number of vertices is $\Delta^2 + 1$. See Figure 1.

By nature, 2-distance colorings and the 2-distance chromatic number of a graph depend a lot on the number of vertices in the neighborhood of every vertex. More precisely, the "sparser" a graph is, the lower its 2-distance chromatic number will be. One way to quantify the sparsity of a graph is through its maximum average degree. The *average degree* ad of a graph $G = (V, E)$ is defined by $\mathrm{ad}(G) = \frac{2|E|}{|V|}$. The *maximum average degree* $\mathrm{mad}(G)$ is the maximum, over all subgraphs H of G, of $\mathrm{ad}(H)$. Another way to measure the sparsity is through the girth, i.e. the length of a shortest cycle. We denote $g(G)$ the girth of G. Intuitively, the higher the girth of a graph is, the sparser it gets. These two measures can actually be linked directly in the case of planar graphs.

A graph is *planar* if one can draw its vertices with points on the plane, and edges with curves intersecting only at its endpoints. When G is a planar graph, Wegner conjectured in 1977 that $\chi^2(G)$ becomes linear in $\Delta(G)$:

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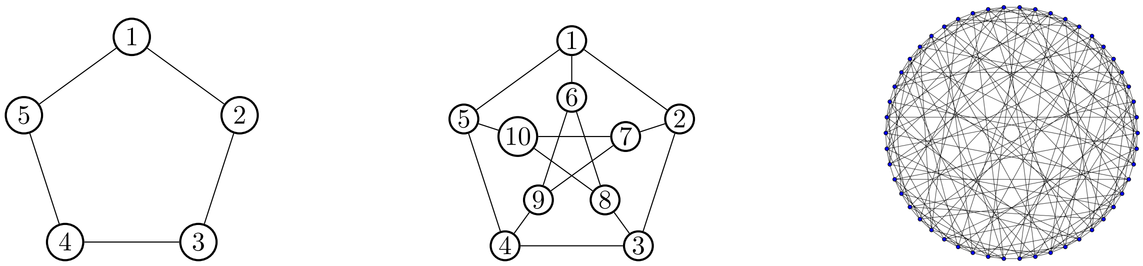


(i) The Moore graph of type (2,2): the odd cycle C_5.

(ii) The Moore graph of type (3,2): the Petersen graph.

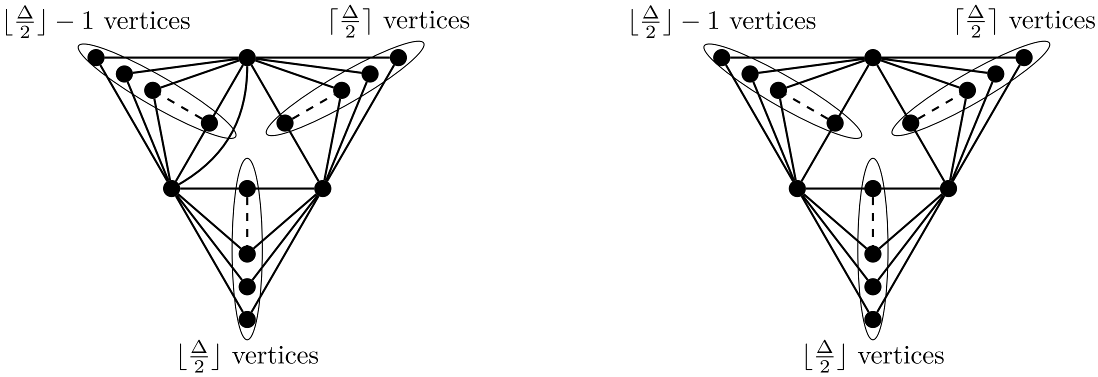
(iii) The Moore graph of type (7,2): the Hoffman-Singleton graph.

Figure 1. Examples of Moore graphs for which $\chi^2 = \Delta^2 + 1$.

Conjecture 1 (Wegner [27])**.** *Let G be a planar graph with maximum degree Δ. Then,*

$$\chi^2(G) \leq \begin{cases} 7, & \text{if } \Delta \leq 3, \\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\ \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

The upper bound for the case where $\Delta \geq 8$ is tight (see Figure 2(i)). Recently, the case $\Delta \leq 3$ was proved by Thomassen [26], and by Hartke *et al.* [16] independently. For $\Delta \geq 8$, Havet *et al.* [17] proved that the bound is $\frac{3}{2}\Delta(1 + o(1))$, where $o(1)$ is as $\Delta \to \infty$ (this bound holds for 2-distance list-colorings). Conjecture 1 is known to be true for some subfamilies of planar graphs, for example K_4-minor free graphs [25].



(i) A graph with girth 3 and $\chi^2 = \lfloor \frac{3\Delta}{2} \rfloor + 1$.

(ii) A graph with girth 4 and $\chi^2 = \lfloor \frac{3\Delta}{2} \rfloor - 1$.

Figure 2. Graphs with $\chi^2 \approx \frac{3}{2}\Delta$.

Wegner's conjecture motivated extensive researches on 2-distance chromatic number of sparse graphs, either of planar graphs with high girth or of graphs with upper bounded maximum average degree which are directly linked due to Proposition 2.

Proposition 2 (Folklore)**.** *For every planar graph G,* $(\text{mad}(G) - 2)(g(G) - 2) < 4$.

As a consequence, any theorem with an upper bound on $\text{mad}(G)$ can be translated to a theorem with a lower bound on $g(G)$ under the condition that G is planar. Many results have taken the following form: *every graph G of* $\text{mad}(G) < m_0$ *and* $\Delta(G) \geq \Delta_0$ *satisfies* $\chi^2(G) \leq \Delta(G) + c(m_0, \Delta_0)$ *where* $c(m_0, \Delta_0)$ *is a constant depending only on* m_0 *and* Δ_0. Due to Proposition 2, as a corollary, the same results on planar graphs of girth $g \geq g_0(m_0)$ where g_0 depends on m_0 follow. Table 1 shows all known such results, up to our knowledge, on the 2-distance chromatic number of planar graphs with fixed girth, either proven directly for planar graphs with high girth or came as a corollary of a result on graphs with bounded maximum average degree.

g_0 \ $\chi^2(G)$	$\Delta+1$	$\Delta+2$	$\Delta+3$	$\Delta+4$	$\Delta+5$	$\Delta+6$	$\Delta+7$	$\Delta+8$
3				$\Delta = 3$ [26, 16]				
4								
5		$\Delta \geq 10^7$ [1][2]	$\Delta \geq 339$ [14]	$\Delta \geq 312$ [13]	$\Delta \geq 15$ [8][1]	$\Delta \geq 12$ [7][2]	$\Delta \neq 7, 8$ [13]	all Δ [12]
6		$\Delta \geq 17$ [3][5]	$\Delta \geq 9$ [7][2]		all Δ [9]			
7	$\Delta \geq 16$ [18][2]	$\Delta \geq 10$[3]	$\Delta \geq 6$ [21][4]	$\Delta = 4$ [10][4]				
8	$\Delta \geq 9$ [24][1]	$\Delta \geq 6$[3]	$\Delta \geq 4$ [21][4]					
9	$\Delta \geq 7$ [23][5]	$\Delta = 5$ [6][4]	$\Delta = 3$ [11][2]					
10	$\Delta \geq 6$ [18][2]							
11		$\Delta = 4$ [10][4]						
12	$\Delta = 5$ [18][2]	$\Delta = 3$ [5][2]						
13								
14	$\Delta \geq 4$ [2][5]							
...								
21	$\Delta = 3$[22]							

Table 1. The latest results with a coefficient 1 before Δ in the upper bound of χ^2.

For example, the result from line "7" and column "$\Delta + 1$" from Table 1 reads as follows : "*every planar graph G of girth at least 7 and of Δ at least 16 satisfies $\chi^2(G) \leq \Delta + 1$*". The crossed out cases in the first column correspond to the fact that, for $g_0 \leq 6$, there are planar graphs G with $\chi^2(G) = \Delta + 2$ for arbitrarily large Δ [4, 15]. The lack of results for $g = 4$ is due to the fact that the graph in Figure 2(ii) has girth 4, and $\chi^2 = \lfloor \frac{3\Delta}{2} \rfloor - 1$ for all Δ.

We are interested in the case $\chi^2(G) \leq \Delta + 2$. In particular, we were looking for the smallest integer Δ_0 such that every graph with maximum degree $\Delta \geq \Delta_0$ and mad $< \frac{8}{3}$ (resp. mad $< \frac{14}{5}$) can be 2-distance colored with $\Delta + 2$ colors. That family contains planar graphs with $\Delta \geq \Delta_0$ and girth at least 8 (resp. 7).

Our main results are the following:

Theorem 3. *If G is a graph with* $\mathrm{mad}(G) \leq \frac{8}{3}$, *then G is 2-distance $(\Delta(G) + 2)$-colorable for $\Delta(G) \geq 6$.*

Theorem 4. *If G is a graph with* $\mathrm{mad}(G) \leq \frac{14}{5}$, *then G is 2-distance $(\Delta(G) + 2)$-colorable for $\Delta(G) \geq 10$.*

For planar graphs, we obtain the following corollaries:

Corollary 5. *If G is a graph with $g(G) \geq 8$, then G is 2-distance $(\Delta(G) + 2)$-colorable for $\Delta(G) \geq 6$.*

Corollary 6. *If G is a graph with $g(G) \geq 7$, then G is 2-distance $(\Delta(G) + 2)$-colorable for $\Delta(G) \geq 10$.*

We will prove Theorems 3 and 4 respectively in Sections 2 and 3 using the same scheme.

2 Proof of Theorem 3

Notations and drawing conventions. For $v \in V(G)$, the *2-distance neighborhood* of v, denoted $N^*_G(v)$, is the set of 2-distance neighbors of v, which are vertices at distance at most two from v, not including v. We also denote $d^*_G(v) = |N^*_G(v)|$. We will drop the subscript and the argument when it is clear from the context. Also for conciseness, from now on, when we say "to color" a vertex, it means to color such vertex differently from all of its colored neighbors at distance at most two. Similarly, any considered coloring will be a 2-distance coloring. We say that a vertex u "sees" a vertex v if $v \in N^*_G(u)$. We also say that u "sees a color" c if there exists $v \in N^*_G(u)$ such that v is colored c.

Some more notations:

- A *d-vertex* (d^+-vertex, d^--vertex) is a vertex of degree d (at least d, at most d). A *$(d \leftrightarrow e)$-vertex* is a vertex of degree between d and e included.
- A *k-path* (k^+-path, k^--path) is a path of length $k+1$ (at least $k+1$, at most $k+1$) where the k internal vertices are 2-vertices. The endvertices of a k-path are 3^+-vertices.
- A *$(k_1, k_2, \ldots, k_d)$-vertex* is a d-vertex incident to d different paths, where the i^{th} path is a k_i-path for all $1 \leq i \leq d$.

[1]Corollaries of more general colorings of planar graphs.
[2]Corollaries of 2-distance list-colorings of planar graphs.
[3] Our results.
[4]Corollaries of 2-distance list-colorings of graphs with a bounded maximum average degree.
[5]Corollaries of 2-distance colorings of graphs with a bounded maximum average degree.

As a drawing convention for the rest of the figures, black vertices will have a fixed degree, which is represented, and white vertices may have a higher degree than what is drawn. Also, we will represented the lower bound on the number of available colors next to each not yet colored vertex in a subgraph H of G when $G - H$ is colored.

Let G_1 be a counterexample to Theorem 3 with the fewest number of vertices. Graph G_1 has maximum degree $\Delta \geq 6$ and $\text{mad}(G) < \frac{8}{3}$. The purpose of the proof is to prove that G_1 cannot exist. In the following we will study the structural properties of G_1. We will then apply a discharging procedure.

2.1 Structural properties of G_1

Lemma 7. *Graph G_1 is connected.*

Proof. Otherwise a component of G_1 would be a smaller counterexample. □

Lemma 8. *The minimum degree of G_1 is at least 2.*

Proof. By Lemma 7, the minimum degree is at least 1. If G_1 contains a degree 1 vertex v, then we can simply remove v and 2-distance color the resulting graph, which is possible by minimality of G_1. Then, we add v back and color it (at most Δ constraints and $\Delta + 2$ colors). □

Lemma 9. *Graph G_1 has no 3^+-paths.*

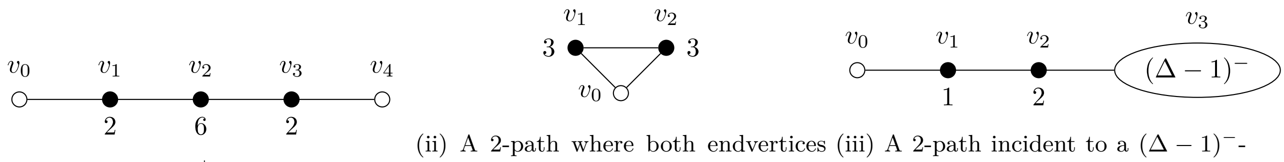
Proof. Suppose G_1 contains a 3^+-path $v_0v_1v_2v_3\dots v_k$ with $k \geq 4$. We color $H = G_1 - \{v_1, v_2, v_3\}$ by minimality of G_1, then we finish by coloring v_1, v_3, and v_2 in this order, which is possible since they have at least respectively 2, 2, and $\Delta \geq 6$ available colors left after the coloring of H. □

Lemma 10. *A 2-path has two distinct endvertices and both have degree Δ.*

Proof. Suppose that G_1 contains a 2-path $v_0v_1v_2v_3$.

If $v_0 = v_3$, then we color $G_1 - \{v_1, v_2\}$ by minimality of G_1 and extend the coloring to G_1 by coloring greedily v_1 and v_2 who has 3 available colors each.

Now, suppose that $v_0 \neq v_3$, and that $d(v_3) \leq \Delta - 1$. We color $G_1 - \{v_1, v_2\}$ by minimality of G_1 and extend the coloring to G_1 by coloring v_1 then v_2, which is possible since they have respectively 1 and 2 available colors left. Thus, $d(v_3) = \Delta$ and the same holds for $d(v_0)$ by symmetry. □



(i) A 3^+-path.

(ii) A 2-path where both endvertices are the same.

(iii) A 2-path incident to a $(\Delta - 1)^-$-vertex.

Figure 3.

Lemma 11. *Graph G_1 has no cycles consisting of 2-paths.*

Proof. Suppose that G_1 contains a cycle consisting of k 2-paths (see Figure 4). We remove all vertices v_{3i+1} and v_{3i+2} for $0 \leq i \leq k-1$. Consider a coloring of the resulting graph. It is then possible to color $v_1, v_2, v_4, \dots, v_{3k-1}$ since each of them has at least two choices of colors (as $d(v_0) = d(v_3) = \dots = d(v_{3(k-1)}) = \Delta$ due to Lemma 10) and by 2-choosability of even cycles. □

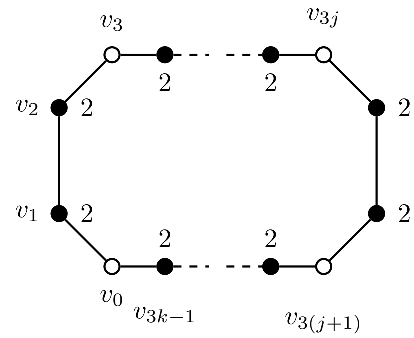


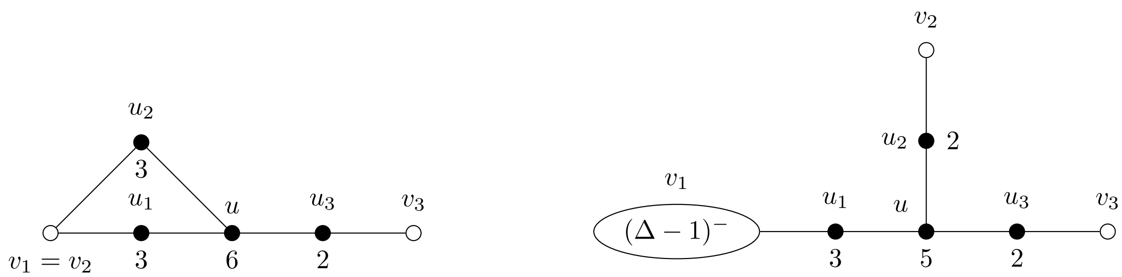
Figure 4. A cycle consisting of consecutive 2-paths.

Lemma 12. *Consider a* $(1,1,1)$*-vertex* u*. The other endvertices of the* 1*-paths incident to* u *are all distincts and are* Δ*-vertices.*

Proof. Suppose there exists a $(1,1,1)$-vertex u with three 2-neighbors u_1, u_2, and u_3. Let v_i be the other endvertex of uu_iv_i for $1 \leq i \leq 3$.

First, suppose by contradiction that $v_1 = v_2$ (and possibly $= v_3$). We color $G_1 - \{u, u_1, u_2, u_3\}$ by minimality of G_1. Then, we color u_3, u_1, u_2, and u in this order, which is possible since they have at least respectively 2, 3, 3, and $\Delta \geq 6$ colors. So, v_1, v_2, and v_3 are all distinct.

Now, suppose w.l.o.g. that $d(v_1) \leq \Delta - 1$ by contradiction. We color $G_1 - \{u, u_1, u_2, u_3\}$ by minimality of G_1. Then, we color u_3, u_2, u_1, and u in this order. So, $d(v_1) = d(v_2) = d(v_3) = \Delta$. □



(i) A $(1,1,1)$-vertex that sees only two vertices at distance 2.

(ii) A $(1,1,1)$-vertex that sees a $(\Delta-1)^-$-vertex at distance 2.

Figure 5.

Definition 13 ($(1,1,1)$-paths)**.** *We call* $v_0v_1v_2v_3v_4$ *a* $(1,1,1)$*-path when* v_0 *and* v_4 *are* Δ*-vertices,* v_1 *and* v_3 *are* 2*-vertices, and* v_2 *is a* $(1,1,1)$*-vertex.*

Lemma 14. *Graph* G_1 *has no cycles consisting of* $(1,1,1)$*-paths.*

Proof. Suppose that G contains a cycle consisting of k $(1,1,1)$-paths (see Figure 6). We remove all vertices v_{4i+1}, v_{4i+2}, v_{4i+3} for $0 \leq i \leq k-1$. Consider a coloring of the resulting graph. We color $v_1, v_3, v_5, \ldots, v_{4k-1}$ since each of them has at least two choices of colors (as $d(v_0) = d(v_4) = \cdots = d(v_{4(k-1)}) = \Delta$ due to Lemma 12) and by 2-choosability of even cycles. Finally, it is easy to color greedily $v_2, v_6, \ldots, v_{4k-2}$ since they each have at most six forbidden colors. □

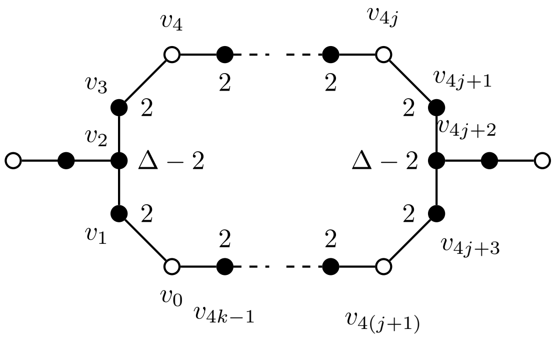


Figure 6. A cycle consisting of consecutive $(1,1,1)$-paths.

Lemma 15. *A $(1,1,0)$-vertex with a $(3 \leftrightarrow \Delta - 3)$-neighbor shares its 2-neighbors with Δ-vertices.*

Proof. Suppose that there exists a $(1,1,0)$-vertex u with a $(3 \leftrightarrow \Delta - 3)$-neighbor. Let u_1 and u_2 be its 2-neighbors. Let $v \neq u$ be the other neighbor of u_1. Suppose w.l.o.g. that $d(v) \leq \Delta - 1$ by contradiction. We color $G_1 - \{u, u_1, u_2\}$ by minimality of G_1. Then, we color u_2, u_1, and u in this order, which is possible since they have at least respectively 1, 2, and 3 colors as we have $\Delta + 2$ colors. □

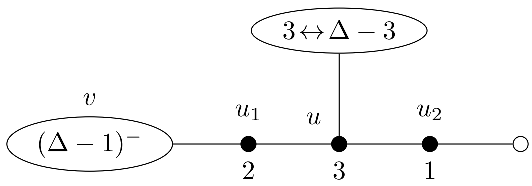


Figure 7. A $(1,1,0)$-vertex with a $(3 \leftrightarrow \Delta - 3)$-neighbor that shares a 2-neighbor with a $(\Delta - 1)^-$-vertex.

Lemma 16. *A Δ-vertex u cannot be incident to a 2-path, a $(1,1,1)$-path, and $\Delta - 2$ other 1^+-paths uu_iv_i ($1 \leq i \leq \Delta - 2$) where each v_i is a 3^--vertex.*

Proof. Let $uu_{\Delta-1}v_{\Delta-1} \notin \{uu_iv_i | 1 \leq i \leq \Delta - 2\}$ be a 1-path where $v_{\Delta-1}$ is a $(1,1,1)$-vertex. Let $uu_\Delta u'_\Delta v_\Delta$ be a 2-path incident to u where $uu_\Delta u'_\Delta \notin \{uu_iv_i | 1 \leq i \leq \Delta - 1\}$. Observe that $v_{\Delta-1} \notin \{v_i | 1 \leq i \leq \Delta - 2\}$ due to Lemma 12 and $v_\Delta \notin \{v_i | 1 \leq i \leq \Delta - 2\}$ due to Lemma 10.

Let $H = u \cup N_G(u) \cup \{u'_\Delta\}$. We color $G - H$ by minimality of G and we uncolor $v_{\Delta-1}$. Let $L(x)$ be the list of remaining colors for a vertex $x \in H \cup \{v_{\Delta-1}\}$. Observe that $|L(u)| \geq \Delta + 2 - (\Delta - 2) \geq 4$, $|L(u'_\Delta)| \geq 2$ (since $d(v_\Delta) = \Delta$ by Lemma 10), $|L(v_{\Delta-1})| \geq \Delta - 2 \geq 4$, $|L(u_i)| \geq \Delta - 1$ (since $d(v_i) \leq 3$) for $1 \leq i \leq \Delta - 2$, $|L(u_{\Delta-1})| \geq \Delta$, and $|L(u_\Delta)| \geq \Delta + 1$. We remove the extra colors from $L(u'_\Delta)$ so that $|L(u'_\Delta)| = 2$. We color u with a color that is not in $L(u'_\Delta)$, then u_1, u_2, …, u_Δ, $v_{\Delta-1}$, and u'_Δ in this order. Observe that when $v_i = v_j$ for $1 \leq i \leq j \leq \Delta - 2$, then $|L(u_i)| \geq \Delta$ and $|L(u_j)| \geq \Delta$ so the order in our coloring still hold. Thus, we obtain a valid coloring of G, which is a contradiction. □

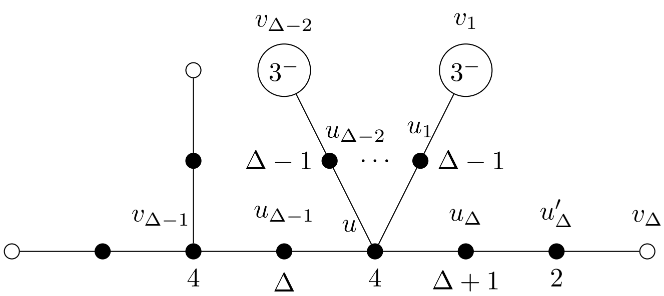


Figure 8. A Δ-vertex incident to a 2-path, a $(1,1,1)$-path, and $\Delta - 2$ other 1^+-paths with 3-endvertices.

2.2 Discharging rules

Definition 17 (2-path sponsors)**.** *Consider the set of* 2*-paths in* G*. By Lemma 10, the endvertices of every* 2*-paths are* Δ*-vertices and by Lemma 11, the graph induced by the edges of all the* 2*-paths of* G *is a forest* $\mathcal{F}$*. For each tree of* $\mathcal{F}$*, we choose one* Δ*-vertex as an arbitrary root. Each* 2*-path is assigned a unique* sponsor *which is the* Δ*-endvertex that is further away from the root. See Figure 9.*

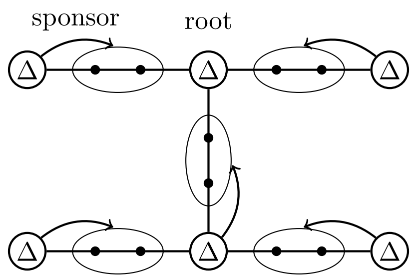


Figure 9. The sponsor assignment in a tree consisting of 2-paths.

Definition 18 ($(1,1,1)$-path sponsors)**.** *Consider the set of* $(1,1,1)$*-paths in* G*. By Lemma 12, the endvertices of every* $(1,1,1)$*-paths are* Δ*-vertices and by Lemma 14, the graph induced by the edges of all the* $(1,1,1)$*-paths of* G *is a forest* $\mathcal{F}$*. For each tree of* $\mathcal{F}$*, we choose one* Δ*-vertex as an arbitrary root. Each* $(1,1,1)$*-vertex* v *is assigned two* sponsors *which are the* Δ*-vertices that are grandsons of* v*. See Figure 10.*

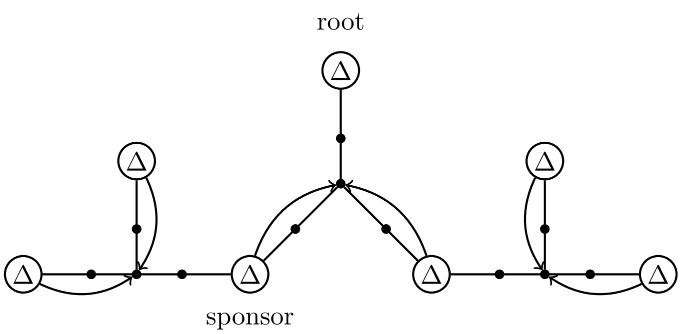


Figure 10. The sponsor assignment in a tree consisting of $(1,1,1)$-paths.

Since we have $\mathrm{mad}(G_1) < \frac{8}{3}$, we must have

$$\sum_{v \in V(G_1)} (3d(v) - 8) < 0 \tag{1}$$

We assign to each vertex v the charge $\mu(v) = 3d(v) - 8$. To prove the non-existence of G_1, we will redistribute the charges preserving their sum and obtaining a non-negative total charge, which will contradict Equation (1).

R0 (see Figure 11): Every 3^+-vertex gives 1 to each 2-neighbor on an incident 1-path.

R1 (see Figure 12): Let u be incident to a 2-path $P = uu_1u_2v$.

- (i) If u is not P's sponsor, then u gives $\frac{3}{2}$ to u_1.
- (ii) If u is P's sponsor, then u gives 2 to u_1 and $\frac{1}{2}$ to u_2.

R2 (see Figure 13): Every 4^+-vertex gives 1 to each 3-neighbor.

R3 (see Figure 14): Let uvw be a 1-path.

- (i) If u is a Δ-vertex, w is a $(1,1,1)$-vertex, and u is w's sponsor, then u gives 1 to w.
- (ii) If u is a Δ-vertex and w is a $(1,1,0)$-vertex, then u gives $\frac{1}{2}$ to w.

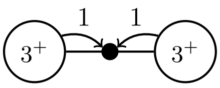
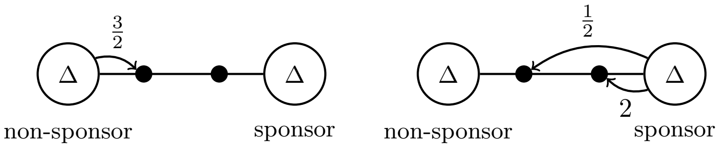


Figure 11. **R0**.



(i) (ii) 2-path sponsor.

Figure 12. **R1**.

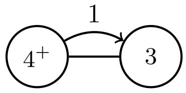
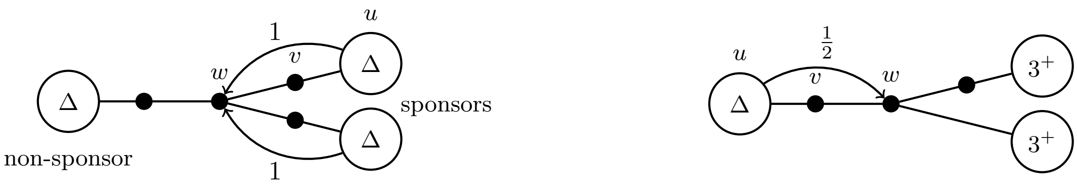


Figure 13. **R2**.



(i) $(1,1,1)$-path sponsor. (ii)

Figure 14. **R3**.

2.3 Verifying that charges on each vertex are non-negative

Let μ^* be the assigned charges after the discharging procedure. In what follows, we prove that:

$$\forall u \in V(G_1), \mu^*(u) \geq 0.$$

Let $u \in V(G_1)$.

Case 1: If $d(u) = 2$, then recall that $\mu(u) = 3 \cdot 2 - 8 = -2$.
There are no 3^+-paths due to Lemma 9 so u must lie on a 1-path or a 2-path.

If u is on a 1-path, then it has two 3^+-neighbors which give it 1 each by **R0**. Thus,

$$\mu^*(u) = -2 + 2 \cdot 1 = 0.$$

If u is on a 2-path, then it either receives 2 from an adjacent sponsor by **R1**(ii), or it receives $\frac{3}{2} + \frac{1}{2} = 2$ from an adjacent non-sponsor Δ-neighbor and a distance 2 sponsor respectively by **R1**(i) and **R1**(ii). Thus,

$$\mu^*(u) = -2 + 2 = 0.$$

Case 2: If $d(u) = 3$, then recall that $\mu(u) = 3 \cdot 3 - 8 = 1$.
Observe that u only gives charge away by **R0** (charge 1 to each 2-neighbor).

If u is a $(1,1,1)$-vertex, then the other endvertices of the 1-paths incident to u are all Δ-vertices due to Lemma 12. Moreover, by Definition 18, u has two sponsors which give it 1 each by **R3**(i). Hence,

$$\mu^*(u) = 1 - 3 \cdot 1 + 2 \cdot 1 = 0.$$

If u is a $(1,1,0)$-vertex with a 4^+-neighbor, then it receives 1 from its neighbor by **R2**. Thus,

$$\mu^*(u) = 1 - 2 \cdot 1 + 1 = 0.$$

If u is a $(1,1,0)$-vertex with a 3-neighbor ($3 \leq \Delta - 3$ since $\Delta \geq 6$), then it receives $\frac{1}{2}$ by **R3**(ii) from each of the other endvertices of its incident 1-paths due to Lemma 15. Thus,

$$\mu^*(u) = 1 - 2 \cdot 1 + 2 \cdot \frac{1}{2} = 0.$$

If u is a $(1^-,0,0)$-vertex, then

$$\mu^*(u) \geq 1 - 1 = 0.$$

Case 3: If $4 \leq d(u) \leq \Delta - 1$, then u only gives away at most 1 to each neighbor by **R0** or **R2**. Thus,

$$\mu^*(u) \geq 3d(u) - 8 - d(u) \geq 2 \cdot 4 - 8 = 0.$$

Case 4: If $d(u) = \Delta$, then we distinguish the following cases.

- If u is neither a 2-path sponsor nor a $(1,1,1)$-path sponsor, then observe that u gives away at most $\frac{3}{2}$ along an incident path by **R1**(i), a combination of **R0** and **R3**(i), or less by **R2**. So at worst,

$$\mu^*(u) \geq 3\Delta - 8 - \frac{3}{2}\Delta \geq \frac{3}{2} \cdot 6 - 8 = 1.$$

- If u is a 2-path sponsor but not a $(1,1,1)$-path sponsor, then u gives $2 + \frac{1}{2} = \frac{5}{2}$ to its unique sponsored incident 2-path by **R1**(ii). For the other incident paths, it gives at most $\frac{3}{2}$ like above. So,

$$\mu^*(u) \geq 3\Delta - 8 - \frac{5}{2} - \frac{3}{2}(\Delta - 1) \geq \frac{3}{2} \cdot 6 - 8 - \frac{5}{2} + \frac{3}{2} = 0.$$

- If u is a $(1,1,1)$-path sponsor but not a 2-path sponsor, then u gives $1 + 1 = 2$ to the unique incident $(1,1,1)$-path containing its assigned $(1,1,1)$-vertex v: 1 to the 2-neighbor by **R0** and 1 to v by **R3**(i). Once again, u gives at most $\frac{3}{2}$ to the other incident paths. So,

$$\mu^*(u) \geq 3\Delta - 8 - 2 - \frac{3}{2}(\Delta - 1) \geq \frac{3}{2} \cdot 6 - 8 - 2 + \frac{3}{2} = \frac{1}{2}.$$

- If u is both a 2-path sponsor and a $(1,1,1)$-path sponsor, then u gives $\frac{5}{2}$ to its unique sponsored 2-path and 2 to its unique assigned $(1,1,1)$-vertex like above.

  Now, let us consider the other $\Delta - 2$ paths incident to u. Observe that when u gives $\frac{3}{2}$ along an incident path either by **R1**(i) or by a combination of **R0** and **R3**(ii), that path must be a 1^+-path where the vertex at distance 2 from u is a 3^--vertex. Due to Lemma 16, u never has to give $\frac{3}{2}$ to each of the $\Delta - 2$ paths. As a result, there exists one path to which u gives at most 1. So at worst,

$$\mu^*(u) \geq 3\Delta - 8 - \frac{5}{2} - 2 - 1 - \frac{3}{2}(\Delta - 3) \geq \frac{3}{2} \cdot 6 - 8 - \frac{5}{2} - 2 - 1 + \frac{9}{2} = 0.$$

We obtain a non-negative amount of charge on each vertex, which is impossible since the total amount of charge is negative. As such, G_1 cannot exist. That concludes the proof of Theorem 3.

3 Proof of Theorem 4

We will reuse similar notations to Section 2. Let G_2 be a counterexample to Theorem 4 with the fewest number of vertices. Graph G_2 has maximum degree $\Delta \geq 10$ and mad $< \frac{14}{5}$. The purpose of the proof is to prove that G_2 cannot exist.

3.1 Structural properties of G_2

Observe that the proofs of Lemmas 7 to 12 and 14 to 16 only rely on the facts that we have a minimal counterexample, two more colors than the maximum degree, and that Δ was large enough ($\Delta(G_1) \geq 6$). All of these still hold for G_2 ($\Delta(G_2) \geq 10$). Thus, we also have the following.

Lemma 19. *Graph G_2 is connected.*

Lemma 20. *The minimum degree of G_2 is at least 2.*

Lemma 21. *Graph G_2 has no 3^+-paths.*

Lemma 22. *A 2-path has two distinct endvertices and both have degree Δ.*

Lemma 23. *Graph G_2 has no cycles consisting of 2-paths.*

Lemma 24. *Consider a $(1,1,1)$-vertex u. The other endvertices of the 1-paths incident to u are all distincts and are Δ-vertices.*

Lemma 25. *Graph G_2 has no cycles consisting of $(1,1,1)$-paths.*

Lemma 26. *A $(1,1,0)$-vertex with a $(3 \leftrightarrow \Delta - 3)$-neighbor shares its 2-neighbors with Δ-vertices.*

Lemma 27. *A Δ-vertex u cannot be incident to a 2-path, a $(1,1,1)$-path, and $\Delta - 2$ other 1^+-paths uu_iv_i ($1 \leq i \leq \Delta - 2$) where each v_i is a 3^--vertex.*

We will show some more reducible configurations.

Lemma 28. *A* $(1, 0, 0)$*-vertex with two* $(3 \leftrightarrow 4)$*-neighbors shares its* 2*-neighbor with a* Δ*-vertex.*

Proof. Suppose by contradiction that there exists a $(1, 0, 0)$-vertex u with two $(3 \leftrightarrow 4)$-neighbors u_1, u_2, and let uvw be the 1-path incident to u, where $d(w) \leq \Delta - 1$. We color $G_2 - \{v\}$ by minimality of G_2, then we uncolor u. Since we have $\Delta + 2 \geq 12$ colors and $d^*(u) = d(u_1) + d(u_2) + 2 \leq 4 + 4 + 2 = 10$, we can always color u last. Finally, v has at least one available color. Thus, we obtain a valid coloring of G_2, which is a contradiction. □

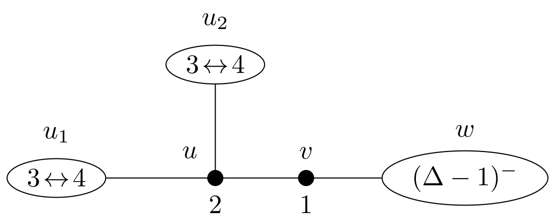


Figure 15. A $(1, 0, 0)$-vertex with two $(3 \leftrightarrow 4)$-neighbor that shares a 2-neighbor with a $(\Delta - 1)^-$-vertex.

Lemma 29. *Consider the four other endvertices of the* 1*-paths incident to a* $(1, 1, 1, 1)$*-vertex. At most one of them is a* $(\Delta - 2)^-$*-vertex.*

Proof. Suppose by contradiction that we have a $(1, 1, 1, 1)$-vertex u incident to four 1-paths uu_iv_i for $1 \leq i \leq 4$, where v_1 and v_2 are $(\Delta - 2)^-$-vertices. We color $G_2 - \{u, u_1, u_2, u_3, u_4\}$ by minimality of G_2. Then, it suffices to color u_3, u_4, u_1, u_2, and u in this order, which is possible since they have at least respectively 2, 2, 4, 4, and 8 available colors as we have $\Delta + 2$ colors and $\Delta \geq 10$. □

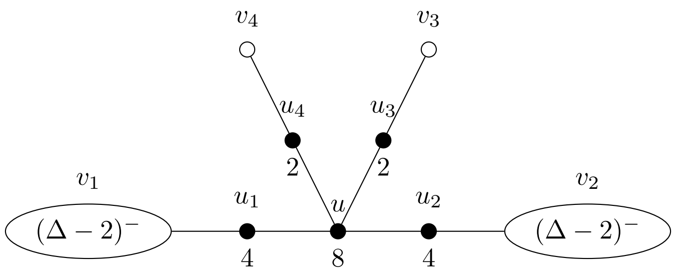


Figure 16. A $(1, 1, 1, 1)$-vertex that sees two $(\Delta - 2)^-$-vertex at distance 2.

3.2 Discharging rules

Since we have $\text{mad}(G_2) < \frac{14}{5}$, we must have

$$\sum_{v \in V(G_2)} (5d(v) - 14) < 0 \tag{2}$$

We assign to each vertex v the charge $\mu(v) = 5d(v) - 14$. To prove the non-existence of G_2, we will redistribute the charges preserving their sum and obtaining a positive total charge, which will contradict Equation (2).

Observe that Definitions 17 and 18 also hold for G_2 thanks to Lemmas 23 and 25.

We apply the following discharging rules:

R0 (see Figure 17): Every 3^+-vertex gives 2 to each 2-neighbor on an incident 1-path.

R1 (see Figure 18): Let u be incident to a 2-path $P = uu_1u_2v$.

(i) If u is not P's sponsor, then u gives $\frac{7}{2}$ to u_1.

(ii) If u is P's sponsor, then u gives 4 to u_1 and $\frac{1}{2}$ to u_2.

R2 (see Figure 19):

(i) Every $(5 \leftrightarrow 7)$-vertex gives 1 to each 3-neighbor.

(ii) Every 8^+-vertex gives 3 to each 3-neighbor.

R3 (see Figure 20): Let uvw be a 1-path.

(i) If u is a Δ-vertex, w is a $(1,1,1)$-vertex, and u is w's sponsor, then u gives 2 to w.

(ii) If u is a Δ-vertex, w is a $(1,1,1)$-vertex, and u is not w's sponsor, then u gives 1 to w.

(iii) If u is a Δ-vertex and w is a $(1,1^-,0)$-vertex, then u gives $\frac{3}{2}$ to w.

(iv) If u is a 9^+-vertex and w is a 4-vertex, then u gives $\frac{2}{3}$ to w.

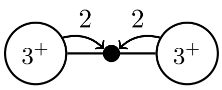
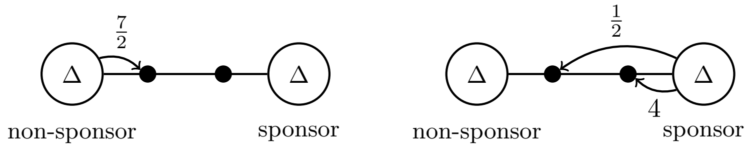


Figure 17. **R0**.

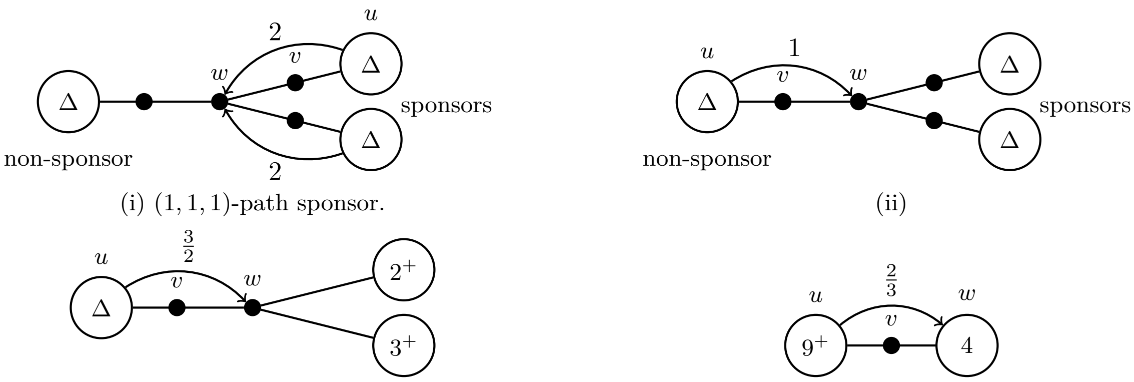


(i) (ii) 2-path sponsor.

Figure 18. **R1**.

(i) (ii)

Figure 19. **R2**.



(i) $(1,1,1)$-path sponsor. (ii)

(iii) (iv)

Figure 20. **R3**.

3.3 Verifying that charges on each vertex are non-negative

Let μ^* be the assigned charges after the discharging procedure. In what follows, we prove that:

$$\forall u \in V(G_2), \mu^*(u) \geq 0.$$

Let $u \in V(G_2)$.

Case 1: If $d(u) = 2$, then recall that $\mu(u) = 5 \cdot 2 - 14 = -4$.
Recall that there exists no 3^+-path due to Lemma 21. So, u must lie on a 1-path or a 2-path.

If u is on a 1-path, then it has two 3^+-neighbors which give it 2 each by **R0**. Thus,

$$\mu^*(u) = -4 + 2 \cdot 2 = 0.$$

If u is on a 2-path, then u receives 4 from an adjacent sponsor by **R1**(ii), or it receives $\frac{7}{2} + \frac{1}{2} = 4$ from an adjacent non-sponsor and a distance 2 sponsor respectively by **R1**(i) and **R1**(ii). Thus,

$$\mu^*(u) = -4 + 4 = 0.$$

Case 2: If $d(u) = 3$, then recall that $\mu(u) = 5 \cdot 3 - 14 = 1$.
Observe that u only gives charge away by **R0** (charge 2 to each 2-neighbor).

If u is a $(1,1,1)$-vertex, then the other endvertices of the 1-paths incident to u are all Δ-vertices due to Lemma 24. As a result, u receives 2 from each of its two sponsors and 1 from the non-sponsor Δ-vertex by **R3**(i) and **R3**(ii). Hence,

$$\mu^*(u) = 1 - 3 \cdot 2 + 2 \cdot 2 + 1 = 0.$$

If u is a $(1,1,0)$-vertex with a 8^+-neighbor, then it receives 3 from its 8^+-neighbor by **R2**(ii). Thus,

$$\mu^*(u) = 1 - 2 \cdot 2 + 3 = 0.$$

If u is a $(1,1,0)$-vertex with an 7^--neighbor ($7 \leq \Delta - 3$ since $\Delta \geq 10$), then it receives $\frac{3}{2}$ by **R3**(iii) from each of the other endvertices of its incident 1-paths due to Lemma 26. Thus,

$$\mu^*(u) = 1 - 2 \cdot 2 + 2 \cdot \frac{3}{2} = 0.$$

If u is a $(1,0,0)$-vertex with a 5^+-neighbor, then it receives at least 1 from that neighbor by **R2**. Thus,

$$\mu^*(u) \geq 1 - 2 + 1 = 0.$$

If u is a $(1,0,0)$-vertex with two $(3 \leftrightarrow 4)$-neighbors, then it receives $\frac{3}{2}$ by **R3**(i) from the other endvertex of its incident 1-path due to Lemma 28. So,

$$\mu^*(u) = 1 - 2 + \frac{3}{2} = \frac{1}{2}.$$

If u is a $(0,0,0)$-vertex, then

$$\mu^*(u) = \mu(u) = 1.$$

Case 3: If $d(u) = 4$, then recall that $\mu(u) = 5 \cdot 4 - 14 = 6$.
Observe that u only gives charge by **R0** (charge 2 to each 2-neighbor).

If u is a $(1,1,1,1)$-vertex, then at least three of the four other endvertices of the 1-paths incident to u are $(\Delta - 1)^+$-vertices (which are 9^+-vertices since $\Delta \geq 10$) due to Lemma 29. As a result, u receives $\frac{2}{3}$ from each of the 9^+-endvertex by **R3**(iv). Hence,

$$\mu^*(u) \geq 6 - 4 \cdot 2 + 3 \cdot \frac{2}{3} = 0.$$

If u is a $(1^-, 1^-, 1^-, 0)$-vertex, then

$$\mu^*(u) \geq 6 - 3 \cdot 2 = 0.$$

Case 3: If $5 \leq d(u) \leq 7$, then u can give 2 to each 2-neighbor by **R0** or 1 to each 3-neighbor by **R2**(i). Thus, at worst we get

$$\mu^*(u) \geq 5d(u) - 14 - 2d(u) \geq 3 \cdot 5 - 14 = 1.$$

Case 4: If $8 \leq d(u) \leq \Delta - 1$, then u gives at most 3 along each incident path by **R2**(ii). Thus, at worst we get

$$\mu^*(u) \geq 5d(u) - 14 - 3d(u) \geq 2 \cdot 8 - 14 = 2.$$

Case 5: If $d(u) = \Delta$, then we distinguish the following cases.

- If u is neither a 2-path sponsor nor a $(1,1,1)$-path sponsor, then observe that u gives away at most $\frac{7}{2}$ along an incident path by **R1**(i) or a combination of **R0** and **R3**(iii). So at worst,

$$\mu^*(u) \geq 5\Delta - 14 - \frac{7}{2}\Delta \geq \frac{3}{2} \cdot 10 - 14 = 1.$$

- If u is a 2-path sponsor but not a $(1,1,1)$-path sponsor, then u gives $4 + \frac{1}{2} = \frac{9}{2}$ to its unique sponsored incident 2-path by **R1**(ii). For the other incident paths, it gives at most $\frac{7}{2}$ like above. So,

$$\mu^*(u) \geq 5\Delta - 14 - \frac{9}{2} - \frac{7}{2}(\Delta - 1) \geq \frac{3}{2} \cdot 10 - 14 - \frac{9}{2} + \frac{7}{2} = 0.$$

- If u is a $(1,1,1)$-path sponsor but not a 2-path sponsor, then u gives $2+2=4$ to the unique incident $(1,1,1)$-path containing its assigned $(1,1,1)$-vertex v: 1 to the 2-neighbor by **R0** and 1 to v by **R3**(i). Once again, u gives at most $\frac{7}{2}$ to the other incident paths. So,

$$\mu^*(u) \geq 5\Delta - 14 - 4 - \frac{7}{2}(\Delta - 1) \geq \frac{3}{2} \cdot 10 - 14 - 4 + \frac{7}{2} = \frac{1}{2}.$$

- If u is both a 2-path sponsor and a $(1,1,1)$-path sponsor, then u gives $\frac{9}{2}$ to its unique sponsored 2-path and 4 to its unique assigned $(1,1,1)$-vertex like above.

  Now, let us consider the other $\Delta - 2$ paths incident to u. Observe that when u gives $\frac{7}{2}$ along an incident path either by **R1**(i) or by a combination of **R0** and **R3**(iii), that path must be a 1^+-path where the vertex at distance 2 from u is a 3^--vertex. Due to Lemma 27, u never has to give $\frac{7}{2}$ to each of the $\Delta - 2$ paths. As a result, there exists one path to which u gives at most 3. So at worst,

$$\mu^*(u) \geq 5\Delta - 14 - \frac{9}{2} - 4 - 3 - \frac{7}{2}(\Delta - 3) \geq \frac{3}{2} \cdot 10 - 14 - \frac{9}{2} - 4 - 3 + \frac{21}{2} = 0.$$

We obtain a non-negative amount of charge on each vertex, which is impossible since the total amount of charge is negative. As such, G_2 cannot exist. That concludes the proof of Theorem 4.

Acknowledgements

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