# 2-distance 4-coloring of planar subcubic graphs with girth at least 21

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**Abstract**

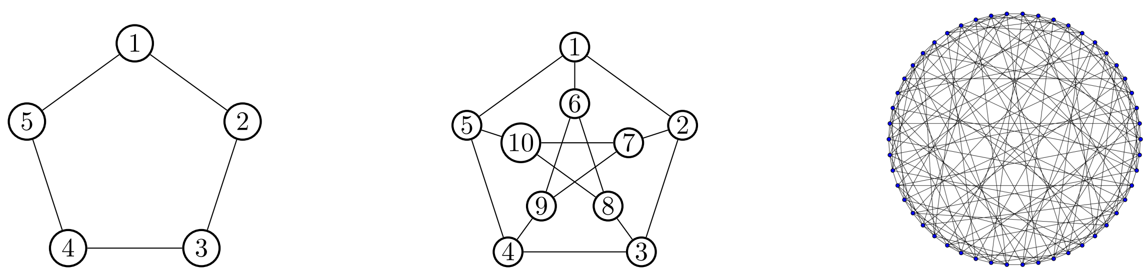
A 2-distance $k$-coloring of a graph is a proper vertex $k$-coloring where vertices at distance at most 2 cannot share the same color. We prove the existence of a 2-distance 4-coloring for planar subcubic graphs with girth at least 21. We also show a construction of a planar subcubic graph of girth 11 that is not 2-distance 4-colorable.

## 1 Introduction

A *$k$-coloring* of the vertices of a graph $G = (V, E)$ is a map $\phi : V \to \{1, 2, \dots, k\}$. A $k$-coloring $\phi$ is a *proper coloring*, if and only if, for all edge $xy \in E, \phi(x) \neq \phi(y)$. In other words, no two adjacent vertices share the same color. The *chromatic number* of $G$, denoted by $\chi(G)$, is the smallest integer $k$ such that $G$ has a proper $k$-coloring. A generalization of $k$-coloring is $k$-list-coloring. A graph $G$ is *$L$-list colorable* if for a given list assignment $L = \{L(v) : v \in V(G)\}$ there is a proper coloring $\phi$ of $G$ such that for all $v \in V(G), \phi(v) \in L(v)$. If $G$ is $L$-list colorable for every list assignment $L$ with $|L(v)| \geq k$ for all $v \in V(G)$, then $G$ is said to be *$k$-choosable* or *$k$-list-colorable*. The *list chromatic number* of a graph $G$ is the smallest integer $k$ such that $G$ is $k$-choosable. List coloring can be very different from usual coloring as there exist graphs with a small chromatic number and an arbitrarily large list chromatic number.

In 1969, Kramer and Kramer introduced the notion of 2-distance coloring [22, 23]. This notion generalizes the "proper" constraint (that does not allow two adjacent vertices to have the same color) in the following way: a *2-distance $k$-coloring* is such that no pair of vertices at distance at most 2 have the same color (similarly to proper $k$-list-coloring, one can also define *2-distance $k$-list-coloring*). The *2-distance chromatic number* of $G$, denoted by $\chi^2(G)$, is the smallest integer $k$ so that $G$ has a 2-distance $k$-coloring.

For all $v \in V$, we denote $d_G(v)$ the degree of $v$ in $G$ and by $\Delta(G) = \max_{v \in V} d_G(v)$ the maximum degree of a graph $G$. For brevity, when it is clear from the context, we will use $\Delta$ (resp. $d(v)$) instead of $\Delta(G)$ (resp. $d_G(v)$). One can observe that, for any graph $G$, $\Delta + 1 \leq \chi^2(G) \leq \Delta^2 + 1$. The lower bound is trivial since, in a 2-distance coloring, every neighbor of a vertex $v$ with degree $\Delta$, and $v$ itself must have a different color. As for the upper bound, a greedy algorithm shows that $\chi^2(G) \leq \Delta^2 + 1$. Moreover, this bound is tight for some graphs, for example, Moore graphs of type $(\Delta, 2)$, which are graphs where all vertices have degree $\Delta$, are at distance at most two from each other, and the total number of vertices is $\Delta^2 + 1$. See Figure 1.




(i) The Moore graph of type (2,2): the odd cycle $C_5$

(ii) The Moore graph of type (3,2): the Petersen graph.

(iii) The Moore graph of type (7,2): the Hoffman-Singleton graph.

Figure 1: Examples of Moore graphs for which $\chi^2 = \Delta^2 + 1$.

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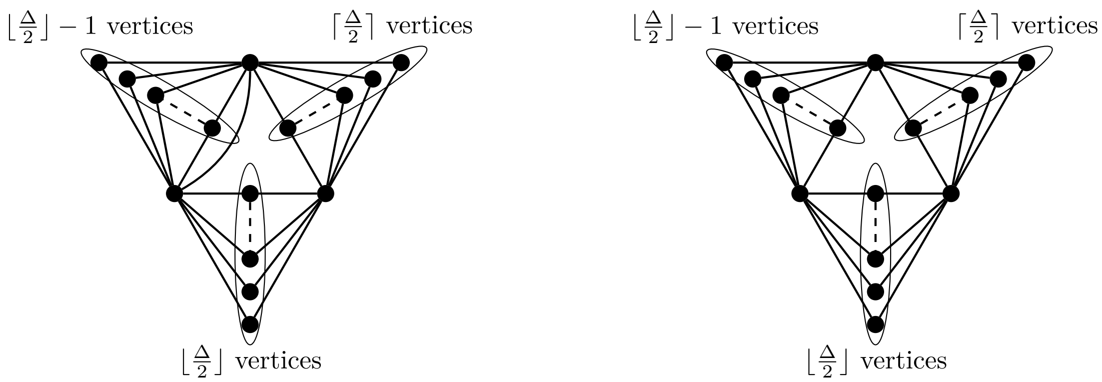
By nature, 2-distance colorings and the 2-distance chromatic number of a graph depend a lot on the number of vertices in the neighborhood of every vertex. More precisely, the "sparser" a graph is, the lower its 2-distance chromatic number will be. One way to quantify the sparsity of a graph is through its maximum average degree. The *average degree* ad of a graph $G = (V, E)$ is defined by $\mathrm{ad}(G) = \frac{2|E|}{|V|}$. The *maximum average degree* $\mathrm{mad}(G)$ is the maximum, over all subgraphs $H$ of $G$, of $\mathrm{ad}(H)$. Another way to measure the sparsity is through the girth, i.e. the length of a shortest cycle. We denote $g(G)$ the girth of $G$. Intuitively, the higher the girth of a graph is, the sparser it gets. These two measures can actually be linked directly in the case of planar graphs.

A graph is *planar* if one can draw its vertices with points on the plane, and edges with curves intersecting only at its endpoints. When $G$ is a planar graph, Wegner conjectured in 1977 that $\chi^2(G)$ becomes linear in $\Delta(G)$:

**Conjecture 1** (Wegner [28])**.** *Let $G$ be a planar graph with maximum degree $\Delta$. Then,*

$$\chi^2(G) \leq \begin{cases} 7, & \text{if } \Delta \leq 3, \\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\ \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

The upper bound for the case where $\Delta \geq 8$ is tight (see Figure 2(i)). Recently, the case $\Delta \leq 3$ was proved by Thomassen [27], and by Hartke *et al.* [19] independently. For $\Delta \geq 8$, Havet *et al.* [20] proved that the bound is $\frac{3}{2}\Delta(1 + o(1))$, where $o(1)$ is as $\Delta \to \infty$ (this bound holds for 2-distance list-colorings). Conjecture 1 is known to be true for some subfamilies of planar graphs, for example $K_4$-minor free graphs [26].




(i) A graph with girth 3 and $\chi^2 = \lfloor \frac{3\Delta}{2} \rfloor + 1$

(ii) A graph with girth 4 and $\chi^2 = \lfloor \frac{3\Delta}{2} \rfloor - 1$.

Figure 2: Graphs with $\chi^2 \approx \frac{3}{2}\Delta$

Wegner's conjecture motivated extensive researches on 2-distance chromatic number of sparse graphs, either of planar graphs with high girth or of graphs with upper bounded maximum average degree which are directly linked due to Proposition 2.

**Proposition 2** (Folklore)**.** *For every planar graph $G$,* $(\mathrm{mad}(G) - 2)(g(G) - 2) < 4$.

As a consequence, any theorem with an upper bound on $\mathrm{mad}(G)$ can be translated to a theorem with a lower bound on $g(G)$ under the condition that $G$ is planar.

Many results have taken the following form: *every graph $G$ of girth $g(G) \leq g_0$ and $\Delta(G) \geq \Delta_0$ satisfies $\chi^2(G) \leq \Delta(G) + c(g_0, \Delta_0)$ where $c(g_0, \Delta_0)$ is a small constant depending only on $g_0$ and $\Delta_0$.* Due to Proposition 2, these type of results sometimes come as a corollary of the same result on graphs with bounded maximum average degree. Table 1 shows all known such results, up to our knowledge, on the 2-distance chromatic number of planar graphs with fixed girth, either proven directly for planar graphs with high girth or came as a corollary of a result on graphs with bounded maximum average degree.

| $\chi^2(G)$ / $g_0$ | $\Delta+1$ | $\Delta+2$ | $\Delta+3$ | $\Delta+4$ | $\Delta+5$ | $\Delta+6$ | $\Delta+7$ | $\Delta+8$ |
|---|---|---|---|---|---|---|---|---|
| 3 | | | | $\Delta = 3$ [27, 19] | | | | |
| 4 | | | | | | | | |
| 5 | | $\Delta \geq 10^7$ [1][2] | $\Delta \geq 339$ [16] | $\Delta \geq 312$ [15] | $\Delta \geq 15$ [10][1] | $\Delta \geq 12$ [9][2] | $\Delta \neq 7, 8$ [15] | all $\Delta$ [14] |
| 6 | | $\Delta \geq 17$ [3][4] | $\Delta \geq 9$ [9][2] | | all $\Delta$ [11] | | | |
| 7 | $\Delta \geq 16$ [21][2] | | | $\Delta = 4$ [12][3] | | | | |
| 8 | $\Delta \geq 9$ [25][1] | | $\Delta = 5$ [8][3] | | | | | |
| 9 | $\Delta \geq 7$ [24][4] | $\Delta = 5$ [8][3] | $\Delta = 3$ [13][2] | | | | | |
| 10 | $\Delta \geq 6$ [21][2] | | | | | | | |
| 11 | | $\Delta = 4$ [12][3] | | | | | | |
| 12 | $\Delta = 5$ [21][2] | $\Delta = 3$ [7][2] | | | | | | |
| 13 | | | | | | | | |
| 14 | $\Delta \geq 4$ [2][4] | | | | | | | |
| ... | | | | | | | | |
| 21 | $\Delta = 3$[5] | | | | | | | |
| 22 | $\Delta = 3$ [6] | | | | | | | |

Table 1: The latest results with a coefficient 1 before $\Delta$ in the upper bound of $\chi^2$.

For example, the result from line "7" and column "$\Delta + 1$" from Table 1 reads as follows : "*every planar graph $G$ of girth at least 7 and of $\Delta$ at least 16 satisfies* $\chi^2(G) \leq \Delta + 1$". The crossed out cases in the first column correspond to the fact that, for $g_0 \leq 6$, there are planar graphs $G$ with $\chi^2(G) = \Delta + 2$ for arbitrarily large $\Delta$ [4, 17]. The lack of results for $g = 4$ is due to the fact that the graph in Figure 2(ii) has girth 4, and $\chi^2 = \lfloor \frac{3\Delta}{2} \rfloor - 1$ for all $\Delta$.

We are interested in the case $\chi^2(G) = \Delta + 1$ as $\Delta + 1$ is a trivial lower bound for $\chi^2(G)$. In particular, we are interested in planar *subcubic* graphs, which are graphs with maximum degree $\Delta = 3$. More precisely, we are trying to answer the following question:

**Question 3.** *What is the smallest $g_0$ such that every planar subcubic graph $G$ with girth $g(G) \geq g_0$ verifies $\chi^2(G) \leq 4$?*

This question was first looked at in [4] by Borodin *et al.* where the authors proved that $g_0 \leq 24$. Later on, Borodin and Ivanova improved the upper bound on $g_0$ to 23 in [5], then 22 in [6]. In this article, we aim to prove that $g_0$ is at most 21.

**Theorem 4.** *If $G$ is a planar subcubic graph with $g(G) \geq 21$, then $\chi^2(G) \leq 4$.*

In Section 2, we present the proof of Theorem 4 using the well-known discharging method. The reducible configurations are obtained by further exploiting the techniques presented in [6].

There was also another approach to Question 3, that is to find lower bounds on $g_0$. While construction of planar graphs with $\chi^2(G) \geq \Delta + 2$ for any $\Delta$ is known for small girth [4, 17]. The first construction with high girth ($g_0 \geq 9$) was presented by Dvořak *et al.* in [18] where the authors relied on an interesting property of 2-distance 4-colorings of vertices at distance 5 from each other. In Section 3, we improve further upon this idea to build a planar subcubic graph of girth 11 with $\chi^2(G) \geq 5$. In other words, we improved the lower bound on $g_0$ from 9 to 11.

## 2 Proof of Theorem 4

**Notations and drawing conventions.** For $v \in V(G)$, the *2-distance neighborhood* of $v$, denoted $N^*_G(v)$, is the set of 2-distance neighbors of $v$, which are vertices at distance at most two from $v$, not including $v$. We also denote $d^*_G(v) = |N^*_G(v)|$. We call $F(G)$ the set of faces of $G$ and for all $f \in F(G)$, $d_G(f)$ is the size of face $f$ (bridges are counted twice). We will drop the subscript and the argument when it is clear from the context. Also for conciseness, from now on, when we say "to color" a vertex, it means to color such vertex differently from all of its colored neighbors at distance at most two. Similarly, any considered coloring will be a 2-distance coloring. We will also say that a vertex $u$ "sees" another vertex $v$ if $u$ and $v$ are at distance at most 2 from each other.

Some more notations:

- A *$d$-vertex* is a vertex of degree $d$.

[1] Corollaries of more general colorings of planar graphs.
[2] Corollaries of 2-distance list-colorings of planar graphs.
[3] Corollaries of 2-distance list-colorings of graphs with a bounded maximum average degree.
[4] Corollaries of 2-distance colorings of graphs with a bounded maximum average degree.
[5] Our result.

- A $k$*-path* ($k^+$-path, $k^-$-path) is a path of length $k+1$ (at least $k+1$, at most $k+1$) where the $k$ internal vertices are 2-vertices and the endvertices are 3-vertices.
- We denote $(k,l,m)$ a 3-vertex incident to a $k$-path, an $l$-path, and an $m$-path.
- A pair of vertices $(k^+,l^+,m)$ and $(m,n^+,p^+)$ joined by an $m$-path will be denoted by $(klm-mnp)$. Similarly, a triple of vertices $u=(k^+,l^+,m)$, $v=(m,n^+,p)$, and $w=(p,q^+,r^+)$ where $u$ and $v$ are joined by an $m$-path and $v$ and $w$ are joined by a $p$-path, will be denoted by $(klm-mnp-pqr)$. This notation is taken from [6].

As a drawing convention for the rest of the figures, black vertices will have a fixed degree, which is represented, and white vertices may have a higher degree than what is drawn.

Let $G$ be a counterexample to Theorem 4 with the fewest number of vertices and edges. Recall that every cycle except $C_5$ is colorable with 4 colors hence, since $G$ has girth at least 21, it has maximum degree $\Delta=3$. The purpose of the proof is to prove that $G$ cannot exist. In the following sections, we will study the structural properties of $G$ (Section 2.2). We will then apply a discharging procedure (Section 2.3).

Due to the Euler formula ($|V|-|E|+|F|=2$), we must have

$$\sum_{u\in V(G)}\left(\frac{19}{2}d(u)-21\right)+\sum_{f\in F(G)}(d(f)-21)=-42<0 \tag{1}$$

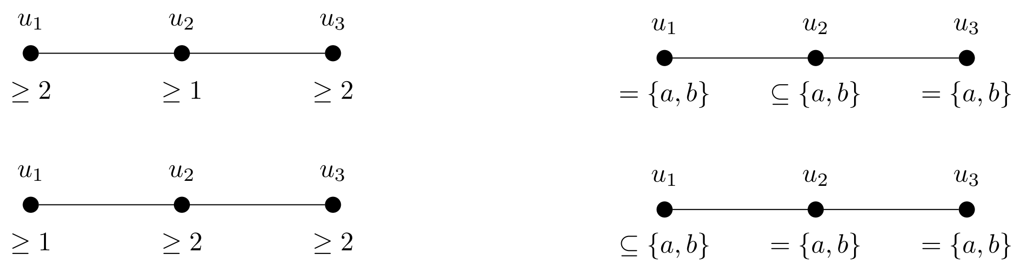
We assign to each vertex $u$ the charge $\mu(u)=\frac{19}{2}d(u)-21$ and to each face $f$ the charge $\mu(f)=d(f)-21$. To prove the non-existence of $G$, we will redistribute the charges preserving their sum and obtaining a non-negative total charge, which will contradict Equation (1).

### 2.1 Useful observations

Before studying the structural properties of $G$, we will introduce some useful observations and lemmas that will be the core of the reducibility proofs of our configurations.

For a vertex $u$, let $L(u)$ denote the set of available colors for $u$ from the set $\{a,b,c,d\}$. For convenience, the lower bound on $|L(u)|$ will be depicted on the figures below the corresponding vertex $u$.

**Lemma 5.** *The graphs depicted in Figure 3 are colorable unless their lists of colors are exactly what is indicated.*




(i) Initial configuration. (ii) Forced list of colors.

Figure 3: An useful non-colorable graph on three vertices.

*Proof.* If $|L(u_1)\cup L(u_2)\cup L(u_3)|\geq 3$, then $u_1$, $u_2$, and $u_3$ are easily colorable (by Hall's theorem by example). Thus, we can assume w.l.o.g. that $L(u_i)\subseteq\{a,b\}$ for all $1\leq i\leq 3$. □

**Lemma 6.** *Let $H$ be a graph on $n\geq 4$ vertices $u_1,u_2,\dots,u_n$. Let the degree and adjacency of $u_1$, $u_2$, and $u_3$ be as depicted in Figure 4. Let $|L(u_1)|\geq 2$, $|L(u_2)|\geq 3$, and $|L(u_3)|\geq d^*_H(u_3)-1$. If, for every $x$ in $L(u_4)$, we have that $u_4,u_5,\dots,u_n$ are colorable with the respective lists $L(u_4)\setminus\{x\},L(u_5),L(u_6),\dots,L(u_n)$, then $H$ is colorable.*

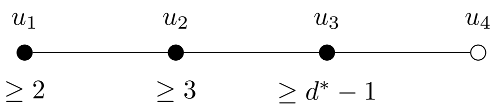



Figure 4: Graph $H$ from Lemma 6.

*Proof.* Suppose by contradiction that $H$ is not colorable. We remove the extra colors from $L(u_1)$ and $L(u_2)$ so that $|L(u_1)| = 2$ and $|L(u_2)| = 3$. We choose $x \in L(u_2) \setminus L(u_1)$. By hypothesis, there exists a coloring of $u_4, u_5, \ldots, u_n$ where $u_4$ is not colored $x$. The remaining vertices, namely $u_1$, $u_2$, and $u_3$ must not be colorable. Since $|L(u_1)| \geq 2$, $|L(u_2)| \geq 3$, and $|L(u_3)| \geq d^*_H(u_3) - 1$, after coloring $u_4, \ldots, u_n$, the lists of available colors for $u_1$, $u_2$, and $u_3$ verify $|L(u_1)| \geq 2$, $|L(u_2)| \geq 2$, and $|L(u_3)| \geq 1$. Since they are not colorable, by Lemma 5, $L(u_1) = L(u_2)$. However, this is impossible since $x \in L(u_2) \setminus L(u_1)$ initially and $x$ remains in $L(u_2)$ since $u_4$ was not colored $x$. □

**Observation 7.** *Lemma 6 means that, by restricting the list $L(u_4)$ to $L(u_4) \setminus \{x\}$ for a well chosen color $x \in L(u_4)$, we can always color $u_1$, $u_2$, and $u_3$ last. As a result, if $H - \{u_1, u_2, u_3\}$ is colorable with $L'(u_4)$ where $|L'(u_4)| = |L(u_4)| - 1$ and $L'(u_4) \subset L(u_4)$ ($L'(u_i) = L(u_i)$ for all $5 \leq i \leq n$), then $H$ is colorable. From now on, for convenience, we will say that we restrict $u_4$ by one color to color $u_1$, $u_2$, and $u_3$ afterwards.*

**Lemma 8.** *The graphs depicted in Figure 5 are all colorable.*

*Proof.* In the following proofs, whenever the size of a list $|L(u)| \geq i$, we assume that $|L(u)| = i$ by removing the extra colors from the list while preserving the inclusions.

(i) If $L(u_1) = L(u_2)$, then we color $u_3$ with a color in $L(u_3) \setminus L(u_2)$, followed by $u_4$, $u_2$, and $u_1$ in this order. If $L(u_1) \neq L(u_2)$, then we color $u_2$ with a color in $L(u_2) \setminus L(u_1)$, followed by $u_4$, $u_3$, and $u_1$ in this order.

(ii) Since $|L(u_1)| \geq 2$, $|L(u'_3)| \geq 3$, and both $L(u_1)$ and $L(u'_3)$ are contained in $\{a, b, c, d\}$, we have a color $x \in L(u_1) \cap L(u'_3)$ by the pigeonhole principle. We color $u_1$ and $u'_3$ with $x$, then $u'_2$, $u_2$, $u_3$, and $u_4$ are colorable by Figure 5i.

(iii) We restrict $u_2$ by one color. Then, we color $u_2$ and $u_1$ in this order first. By Lemma 6, we color $u_3$, $u_4$, and $u_5$ last.

(iv) If $L(u_1)$ and $L(u'_3)$ share a common color $x$, then we color $u_1$ and $u'_3$ with $x$. The remaining vertices $u_5$, $u_4$, $u_3$, and $u_2$ are colorable by Figure 5i. So, $L(u_1) \cap L(u'_3) = \emptyset$ and by symmetry, we also have $L(u_5) \cap L(u'_3) = \emptyset$.

W.l.o.g. we set $L(u'_3) = \{a, b\}$. As a result, $L(u_1) = L(u_5) = \{c, d\}$. Recall that $L(u_1) \subseteq L(u_2)$ and $L(u_5) \subseteq L(u_4)$. So, we can color $u_1$ and $u_4$ with $c$, $u_2$ and $u_5$ with $d$, $u_3$ with $a$, and $u'_3$ with $b$.

(v) We have $L(u'_3) \subseteq L(u_3)$. Otherwise, we can color $u'_3$ with a color in $L(u'_3) \setminus L(u_3)$, then $u_1$, $u_2$, $u_3$, $u_4$, and $u_5$ are colorable by Figure 5iii.

If $L(u_1)$ and $L(u'_3)$ share a common color $x$, then we color $u_1$ and $u'_3$ with $x$, and $u_2$, $u_3$, $u_4$, and $u_5$ are colorable by Figure 5i.

If $L(u_1) \cap L(u'_3) = \emptyset$, then w.l.o.g. we set $L(u_1) = \{a, b\}$ and $L(u'_3) = \{c, d\}$. Since $|L(u_2)| \geq 3$, w.l.o.g. we color $u_2$ with $a$ then $u_1$ with $b$. As both $L(u'_3)$ and $L(u_3)$ contain $\{c, d\}$, we still have $|L(u'_3)| \geq 2$ and $|L(u_3)| \geq 2$, thus $u'_3$, $u_3$, $u_4$, and $u_5$ are colorable by Figure 5i.

(vi) By the pigeonhole principle, there exists $x \in L(u'_3) \cap L(u_5)$. If $x \notin L(u_2)$, then we color $u'_3$ and $u_5$ with $x$. The remaining vertices $u_1$, $u_2$, $u_3$, and $u_4$ are colorable by Figure 5i. So $x \in L(u_2)$.

We also have $L(u_1) = L(u_2)$. Otherwise, we color $u_2$ with a color in $L(u_2) \setminus L(u_1)$, then $u_5$, $u_4$, $u_3$, $u'_3$ are colorable by Figure 5i, and we finish by coloring $u_1$.

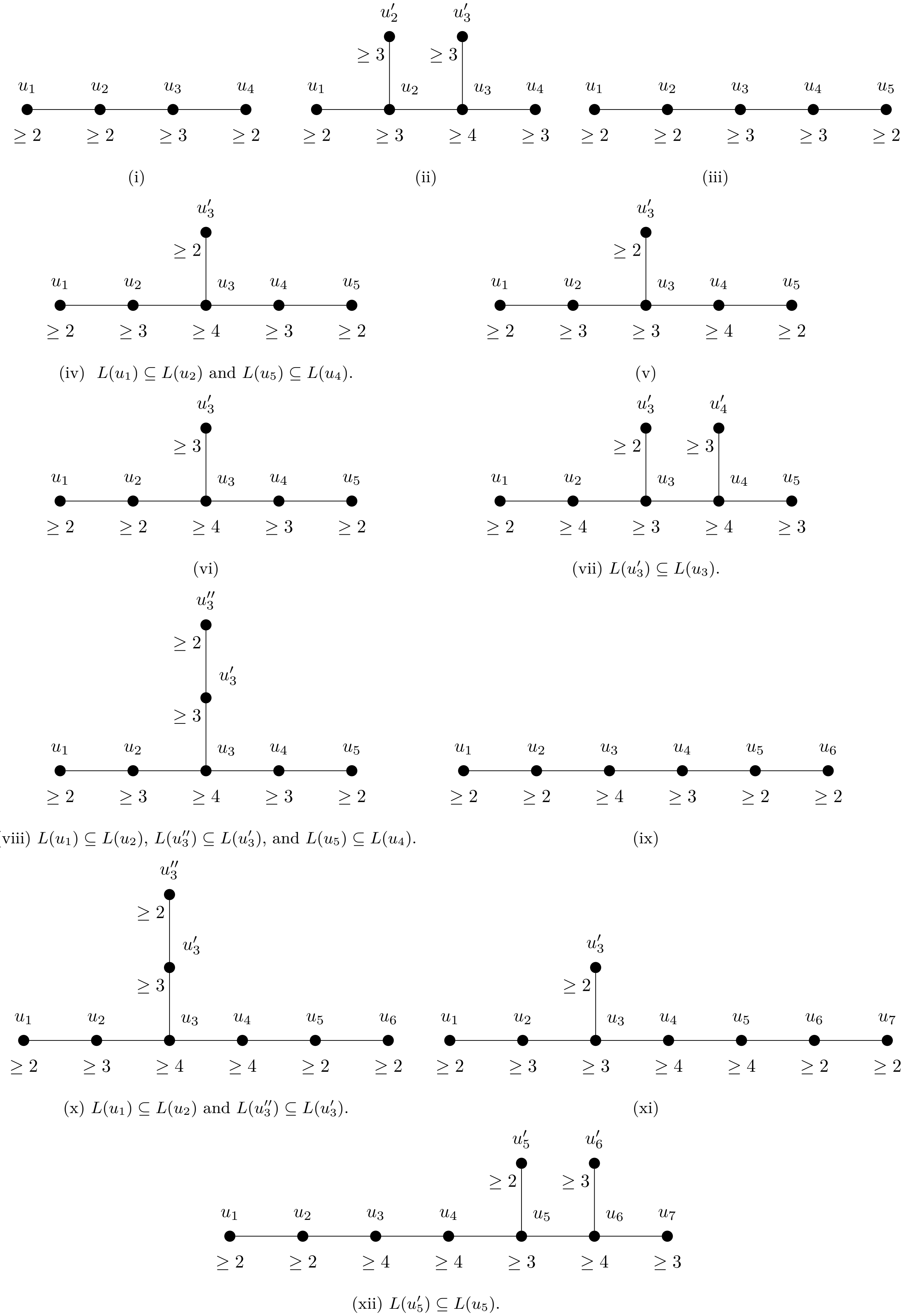
Since $x \in L(u'_3) \cap L(u_5) \cap L(u_2) \cap L(u_1)$, we color $u_1$, $u'_3$, and $u_5$ with $x$, then we color $u_2$, $u_4$, and $u_3$ in this order.

(vii) If there exists $x \in L(u_3) \setminus L(u_5)$, then we color $u_3$ with $x$, then $u'_3$, $u_1$, $u_2$, $u_4$, $u'_4$, and $u_5$ in this order.

If $L(u_3) = L(u_5)$, then we color $u_4$ with a color $y$ in $L(u_4) \setminus L(u_5)$. Recall that $L(u'_3) \subseteq L(u_3)$, so $y \notin L(u'_3) \cup L(u_3) \cup L(u_5)$. We color $u_1$, $u_2$, $u_3$, and $u'_3$ by Figure 5i. Finally, we finish by color $u'_4$ and $u_5$ in this order.

(viii) If there exists two same sets of colors between $L(u_2)$, $L(u_4)$, and $L(u'_3)$, say $L(u_2) = L(u_4)$, then we color $u_3$ with $x \in L(u_3) \setminus L(u_2)$. Recall that $L(u_1) \subseteq L(u_2)$ and $L(u_5) \subseteq L(u_4)$ so $x \notin L(u_1) \cup L(u_2) \cup L(u_4) \cup L(u_5)$. We finish by coloring $u''_3$, $u'_3$, $u_1$, $u_2$, $u_4$, $u_5$ in this order.

If $L(u_2)$, $L(u_4)$, and $L(u'_3)$ are all different, then we color the graph as follows. By the pigeonhole principle, two sets between $L(u_1)$, $L(u_5)$, and $L(u''_3)$ must share a common color, say $L(u_1) \cap L(u_5) \neq \emptyset$. In other words, $|L(u_1) \cup L(u_5)| \leq 3$. Then, we color $u_3$ with a color in $L(u_3) \setminus (L(u_1) \cup L(u_5))$. We color $u''_3$ and $u'_3$ in this order. Now, we can color $u_2$ and $u_4$ since they see the same two colors but initially $L(u_2) \neq L(u_4)$. Finally, we finish by coloring $u_1$ and $u_5$.



(i)

(ii)

(iii)

(iv) $L(u_1) \subseteq L(u_2)$ and $L(u_5) \subseteq L(u_4)$.

(v)

(vi)

(vii) $L(u_3') \subseteq L(u_3)$.

(viii) $L(u_1) \subseteq L(u_2)$, $L(u_3'') \subseteq L(u_3')$, and $L(u_5) \subseteq L(u_4)$.

(ix)

(x) $L(u_1) \subseteq L(u_2)$ and $L(u_3'') \subseteq L(u_3')$.

(xi)

(xii) $L(u_5') \subseteq L(u_5)$.

Figure 5: Colorable graphs.

(ix) If $L(u_1) = L(u_2)$, then we restrict $L(u_3)$ to $L(u_3) \setminus L(u_2)$. We color $u_3$, $u_4$, $u_5$, $u_6$ by Figure 5i, then we finish by coloring $u_2$ and $u_1$ in this order.

If there exists $x \in L(u_1) \setminus L(u_2)$, then we color $u_1$ with $x$. Finally, $u_6$, $u_5$, $u_4$, $u_3$, and $u_2$ are colorable by Figure 5iii.

(x) If $L(u_5) = L(u_6)$, then we restrict $L(u_4)$ to $L(u_4) \setminus L(u_5)$. Recall that we have $L(u_1) \subseteq L(u_2)$ and $L(u_3'') \subseteq L(u_3')$. We can thus color $u_1$, $u_2$, $u_3$, $u_4$, $u_3'$, and $u_3''$ by Figure 5iv. We finish by coloring $u_5$ and $u_6$ in this order.

If there exists $a \in L(u_6) \setminus L(u_5)$, then we color $u_6$ with $a$. Observe that $L(u_5) \subseteq \{b, c, d\} = L(u_4)$ after we color $u_6$ with $a$. Recall that we also have $L(u_1) \subseteq L(u_2)$ and $L(u_3'') \subseteq L(u_3')$. So, we color the remaining vertices $u_1$, $u_2$, $u_3$, $u_3'$, $u_3''$, $u_4$, and $u_5$ by Figure 5viii

(xi) If $L(u_6) = L(u_7)$, then we restrict $L(u_5)$ to $L(u_5) \setminus L(u_6)$. We color $u_1$, $u_2$, $u_3$, $u_3'$, $u_4$, and $u_5$ by Figure 5v. We finish by coloring $u_6$ and $u_7$ in this order.

If there exists $x \in L(u_7) \setminus L(u_6)$, then we color $u_7$ with $x$. We restrict $u_3$ by one color to color $u_4$, $u_5$, and $u_6$ last by Lemma 6. Then, $u_3'$, $u_3$, $u_2$, and $u_1$ are colorable by Figure 5i.

(xii) If $L(u_1) = L(u_2)$, then we restrict $L(u_3)$ to $L(u_3) \setminus L(u_2)$. We color $u_3$, $u_4$, $u_5$, $u_5'$, $u_6$, $u_6'$, and $u_7$ by Figure 5vii. Then, we finish by coloring $u_2$ and $u_1$ in this order.

If there exists $x \in L(u_2) \setminus L(u_1)$, then we color $u_2$ with $x$. We color $u_5'$, $u_5$, $u_4$, $u_6$, $u_6'$, and $u_7$ by Figure 5ii. Finally, we finish by coloring $u_3$ and $u_1$ in this order.

□

## 2.2 Structural properties of $G$

**Lemma 9.** *Graph $G$ is connected.*

Otherwise a component of $G$ would be a smaller counterexample.

**Lemma 10.** *The minimum degree of $G$ is at least 2.*

By Lemma 9, the minimum degree is at least 1 or $G$ would be a single isolated vertex which is 4-colorable. If $G$ contains a degree 1 vertex $v$, then we can simply remove the unique edge incident to $v$ and 2-distance color the resulting graph, which is possible by minimality of $G$. Then, we add the edge back and color $v$ (at most 3 constraints and 4 colors).

**Lemma 11** ([6] Lemmas 10,11, and 12)**.** *Graph $G$ has no:*

*(i)* $6^+$*-paths*

*(ii)* $(1^+, 4^+, 5^+)$

*(iii)* $(2^+, 3^+, 4^+)$

*(iv)* $(3^+, 3^+, 3^+)$

*(v)* $(330 - 045)$

*(vi)* $(431 - 133)$

The proofs of the reducibility of these configurations are presented in [6] with the same notations. These configurations were reduced for planar subcubic graphs of girth at least 22 where all 3-vertices and 2-vertices on the incident paths are distinct, but the same proofs hold for $G$ since the girth is still high enough for all vertices to remain distinct.

The following configurations are new or stronger versions of configurations in [6].

**Lemma 12.** *Graph $G$ cannot contain the following pairs:*

*(i)* $(430 - 024)$

*(ii)* $(540 - 014)$

*(iii)* $(431 - 114)$

*(iv)* $(422 - 223)$

*(v)* $(422 - 214)$

*(vi)* $(412 - 233)$

*(vii)* $(332 - 233)$

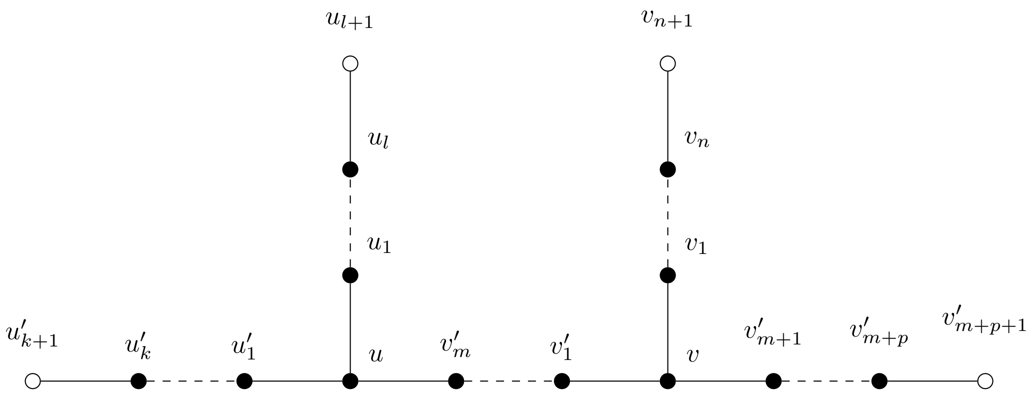



Figure 6: Lemma 12 notations.

*Proof.* First, we define the following notations:

- Let $u = (k^+, l^+, m)$ and $v = (m, n^+, p^+)$ form the pair $(klm - mnp)$.
- Let $uu'_1u'_2 \dots u'_{k+1}$ be the $k^+$-path incident to $u$.
- Let $uu_1u_2 \dots u_{l+1}$ be the $l^+$-path incident to $u$.
- Let $vv'_1v'_2 \dots v'_m u$ be the $m$-path incident to $u$ and $v$.
- Let $vv_1v_2 \dots v_{n+1}$ be the $n^+$-path incident to $v$.
- Let $vv'_{m+1}v'_{m+2} \dots v'_{m+p+1}$ be the $p^+$-path incident to $v$.
- For every pair $(klm - mnp)$ from (i) to (vii), we define the subgraph

  $H = \{u, v, u'_1, u'_2, \dots, u'_{k-1}, u_1, u_2, \dots, u_{l-1}, v'_1, v'_2, \dots, v'_{m+p-1}, v_1, v_2, \dots, v_{n-1}\}$.

First, observe that all vertices in $H$ are distinct since $G$ has girth at least 21. In the following proofs, we will always color $G - H$ first, which is possible by minimality of $G$. For each vertex of $H$, its list of available colors will always be $\{a, b, c, d\}$ from which we removed the colors it sees on its neighbors from $G - H$. Then, we will show that the coloring of $G - H$ is extendable to $H$ using colorable graphs from Lemma 8. For convenience, we will cite Figure 5 from now on.

Also observe that when two adjacent vertices $x_1$, $x_2$ in $H$ sees a common color with $|L(x_1)| \leq |L(x_2)|$, then $L(x_1) \subseteq L(x_2)$. This simple remark will be used throughout the proofs, mostly to justify the use of Figure 5(iv), (vii), (viii), (x), and (xii). For conciseness, we will state the inclusions directly when needed.

(i) We restrict $u$ by one color to color $u'_1$, $u'_2$, and $u'_3$ last. We restrict $v$ by one color to color $v'_1$, $v'_2$, and $v'_3$ afterwards. Finally, $v_1$, $v$, $u$, $u_1$, and $u_2$ are colorable by Figure 5iii.

(ii) We restrict $v$ by one color to color $v'_1$, $v'_2$, and $v'_3$ last. We restrict $u'_1$ by one color to color $u'_2$, $u'_3$, and $u'_4$ afterwards. Finally, we color $v$, then $u'_1$, $u$, $u_1$, $u_2$, and $u_3$ are colorable by Figure 5iii.

(iii) We restrict $u$ by one color to color $u'_1$, $u'_2$, and $u'_3$ last. We restrict $v$ by one color to color $v'_2$, $v'_3$, and $v'_4$ afterwards. Finally, we color $v$, then $v'_1$, $u$, $u_1$, and $u_2$ are colorable by Figure 5i.

(iv) We restrict $u$ by one color to color $u'_1$, $u'_2$, and $u'_3$ last. Then, $v'_4$, $v'_3$, $v$, $v_1$, $v'_1$, $v'_2$, $u$, and $u_1$ are colorable by Figure 5xi.

(v) We restrict $v$ by one color to color $v'_3$, $v'_4$, and $v'_5$ last. We restrict $u$ by one color to color $u'_1$, $u'_2$, and $u'_3$ afterwards. Finally, we color $v$, then $u_1$, $u$, $v'_2$, and $v'_1$ are colorable by Figure 5i.

(vi) We restrict $u$ by one color to color $u'_1$, $u'_2$, and $u'_3$ last. Then, we color $u$ and observe that since $L(v'_2) \subseteq L(v'_1)$, $L(v_2) \subseteq L(v_1)$, and $L(v'_4) \subseteq L(v'_3)$, $v'_2$, $v'_1$, $v$, $v_1$, $v_2$, $v'_3$, and $v'_4$ are colorable by Figure 5viii.

(vii) We color $v$ with $x \in L(v) \setminus L(v_1)$. Observe that $L(v_2) \subseteq L(v_1)$ so $x \notin L(v_1) \cup L(v_2)$. Then, we color $v'_4$, and $v'_3$ in this order. Since $L(u'_2) \subseteq L(u'_1)$, $L(u_2) \subseteq L(u_1)$, and $L(v'_1) \subseteq L(v'_2)$, $u'_2$, $u'_1$, $u$, $u_1$, $u_2$, $v'_2$ and $v'_1$ are colorable by Figure 5viii. Finally, we finish by coloring $v_1$ and $v_2$.

□

**Lemma 13.** *Graph $G$ cannot contain the following triples:*

*(i)* $(550 - 020 - 045)$

*(ii)* $(440 - 040 - 024)$

*(iii)* $(550 - 021 - 134)$

*(iv)* $(420 - 031 - 134)$

*(v)* $(550 - 022 - 224)$

*(vi)* $(540 - 032 - 214)$

*(vii)* $(540 - 032 - 233)$

*(viii)* $(420 - 042 - 214)$

*(ix)* $(420 - 042 - 233)$

*(x)* $(431 - 131 - 124)$

*(xi)* $(421 - 141 - 124)$

*(xii)* $(431 - 112 - 224)$

*(xiii)* $(421 - 132 - 233)$

*(xiv)* $(421 - 132 - 214)$

*(xv)* $(422 - 222 - 214)$

*(xvi)* $(332 - 222 - 224)$

*(xvii)* $(332 - 232 - 233)$

*(xviii)* $(332 - 232 - 214)$

*(xix)* $(412 - 232 - 214)$

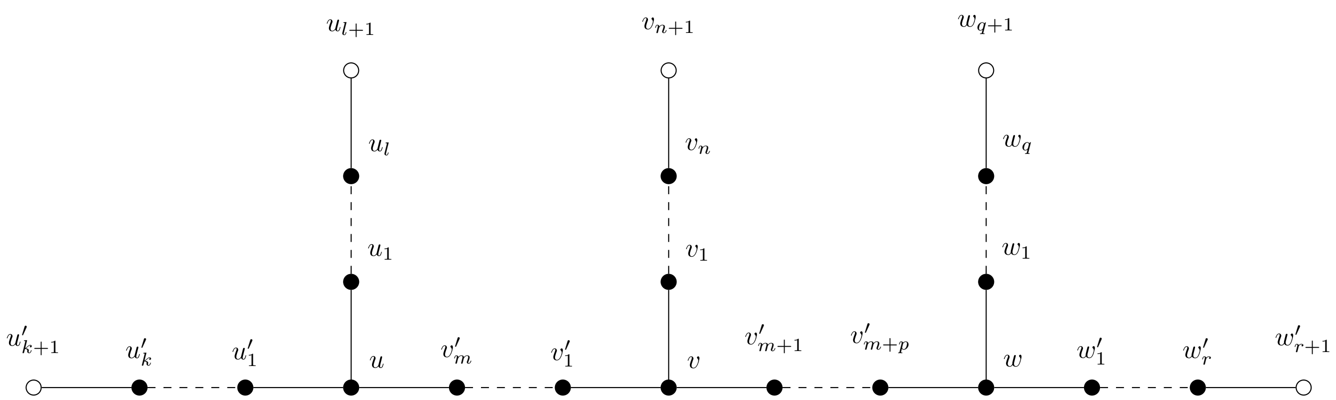



Figure 7: Lemma 13 notations.

*Proof.* We will use similar notations to the proofs of Lemma 12:

- Let $u = (k^+, l^+, m)$, $v = (m, n^+, p)$, and $w = (p, q^+, r^+)$ form the triple $(klm - mnp - pqr)$.
- Let $uu'_1u'_2 \dots u'_{k+1}$ be the $k^+$-path incident to $u$.
- Let $uu_1u_2 \dots u_{l+1}$ be the $l^+$-path incident to $u$.
- Let $vv'_1v'_2 \dots v'_m u$ be the $m$-path incident to $u$ and $v$.
- Let $vv_1v_2 \dots v_{n+1}$ be the $n^+$-path incident to $v$.
- Let $vv'_{m+1}v'_{m+2} \dots v'_{m+p}w$ be the $p$-path incident to $v$ and $w$.
- Let $ww_1w_2 \dots w_{q+1}$ be the $q^+$-path incident to $w$.

- Let $ww'_1w'_2\ldots w'_{r+1}$ be the $r^+$-path incident to $u$.
- For every triple $(klm - mnp - pqr)$ from (i) to (xix), we define the subgraph

  $H = \{u, v, w, u'_1, u'_2, \ldots, u'_{k-1}, u_1, u_2, \ldots, u_{l-1}, v'_1, v'_2, \ldots, v'_{m+p}, v_1, v_2, \ldots, v_{n-1}, w_1, w_2, \ldots, w_{q-1}, w'_1, w'_2, \ldots, w'_{r-1}\}$.

Similarly, all vertices in $H$ are distinct since $G$ has girth at least 21. We will color $G - H$ by minimality of $G$ first, then extend that coloring to $H$ using Figure 5.

(i) We restrict $u'_1$ by one color to color $u'_2$, $u'_3$, and $u'_4$ last. We restrict $u_1$ by one color to color $u_2$, $u_3$, and $u_4$ afterwards. We restrict $w$ by one color to color $w_1$, $w_2$, and $w_3$ afterwards. We restrict $w'_1$ by one color to color $w'_2$, $w'_3$, and $w'_4$ afterwards. Now, we color $v_1$, $v$, $w$, $u$, $u_1$, and $u'_1$ by Figure 5ii. Then, we color $w'_1$.

(ii) We restrict $u$ by one color to color $u'_1$, $u'_2$, and $u'_3$ last. We restrict $u$ again by one color to color $u_1$, $u_2$, and $u_3$ afterwards. We restrict $v$ by one color to color $v_1$, $v_2$, and $v_3$ afterwards. We restrict $w$ by one color to color $w'_1$, $w'_2$, and $w'_3$ afterwards. We color the remaining vertices $w_1$, $w$, $v$, and $u$ by Figure 5i.

(iii) We restrict $u'_1$ by one color to color $u'_2$, $u'_3$, and $u'_4$ last. We restrict $u_1$ by one color to color $u_2$, $u_3$, and $u_4$ afterwards. We restrict $w$ by one color to color $w'_1$, $w'_2$, and $w'_3$ afterwards. We restrict $v'_1$ by one color to color $w$, $w_1$, and $w_2$ afterwards. The remaining vertices $v_1$, $v$, $v'_1$, $u$, $u_1$, and $u'_1$ are colorable by Figure 5ii.

(iv) We restrict $u$ by one color to color $u'_1$, $u'_2$, and $u'_3$ last. We restrict $w$ by one color to color $w'_1$, $w'_2$, and $w'_3$ afterwards. We restrict $v'_1$ by one color to color $w$, $w_1$, and $w_2$ afterwards. The remaining vertices $u_1$, $u$, $v$, $v'_1$, $v_1$, and $v_2$ are colorable by Figure 5vi.

(v) We restrict $u'_1$ by one color to color $u'_2$, $u'_3$, and $u'_4$ last. We restrict $u_1$ by one color to color $u_2$, $u_3$, and $u_4$ afterwards. We restrict $w$ by one color to color $w'_1$, $w'_2$, and $w'_3$ afterwards. The remaining vertices $w_1$, $w$, $v'_2$, $v'_1$, $v$, $v_1$, $u$, $u'_1$, and $u_1$ are colorable by Figure 5xii as $L(v_1) \subseteq L(v)$.

(vi) We restrict $u'_1$ by one color to color $u'_2$, $u'_3$, and $u'_4$ last. We restrict $w$ by one color to color $w'_1$, $w'_2$, and $w'_3$ afterwards. We color $v$ with $x \in L(v) \setminus L(v_1)$. Observe that $L(v_2) \subseteq L(v_1)$ so $x \notin L(v_1) \cup L(v_2)$. Now, we color $w$, $v'_2$, $v'_1$ in this order. The vertices $u'_1$, $u$, $u_1$, $u_2$, and $u_3$ are colorable by Figure 5iii. Then, we color the remaining vertices $v_1$ and $v_2$ in this order.

(vii) We restrict $u'_1$ by one color to color $u'_2$, $u'_3$, and $u'_4$ last. We color $w$ with $x \in L(w) \setminus L(w_1)$. Observe that $L(w_2) \subseteq L(w_1)$ so $x \notin L(w_1) \cup L(w_2)$. We color $v$ with $y \in L(v) \setminus L(v_1)$. Observe that $L(v_2) \subseteq L(v_1)$ so $y \notin L(v_1) \cup L(v_2)$. Now, we color $w'_2$, $w'_1$, $v'_2$, $v'_1$ in this order. The vertices $u'_1$, $u$, $u_1$, $u_2$, and $u_3$ are colorable by Figure 5iii. Then, we color the remaining vertices $v_1$, $v_2$, $w_1$, and $w_2$ in this order.

(viii) We restrict $u$ by one color to color $u'_1$, $u'_2$, and $u'_3$ last. We restrict $w$ by one color to color $w'_1$, $w'_2$, and $w'_3$ afterwards. We restrict $v$ by one color to color $v_1$, $v_2$, and $v_3$ afterwards. We color $w$ then the remaining vertices $u_1$, $u$, $v$, $v'_1$, and $v'_2$ are colorable by Figure 5iii.

(ix) We restrict $u$ by one color to color $u'_1$, $u'_2$, and $u'_3$ last. We restrict $v$ by one color to color $v_1$, $v_2$, and $v_3$ afterwards. We color $w$ with $x \in L(w) \setminus L(w_1)$. Observe that $L(w_2) \subseteq L(w_1)$ so $x \notin L(w_1) \cup L(w_2)$. We color $w'_2$ and $w'_1$ in this order. The vertices $u_1$, $u$, $v$, $v'_1$, and $v'_2$ are colorable by Figure 5iii. Now, we color the remaining vertices $w_1$ and $w_2$ in this order.

(x) We restrict $u$ by one color to color $u'_1$, $u'_2$, and $u'_3$ last. We restrict $v'_1$ by one color to color $u$, $u_1$, and $u_2$ afterwards. We restrict $w$ by one color to color $w'_1$, $w'_2$, and $w'_3$ afterwards. We restrict $L(v'_2)$ to $L(v'_2) \setminus L(w_1)$. We color the vertices $w$, $v'_2$, $v$, $v'_1$, $v_1$ and $v_2$ by Figure 5vi. Then, we color the remaining vertex $w_1$.

(xi) We restrict $u$ by one color to color $u'_1$, $u'_2$, and $u'_3$ last. We restrict $v$ by one color to color $v_1$, $v_2$, and $v_3$ afterwards. We restrict $w$ by one color to color $w'_1$, $w'_2$, and $w'_3$ afterwards. We restrict $L(v'_1)$ to $L(v'_1) \setminus L(u_1)$. We color $w_1$, $w$, $v'_2$, $v$, $v'_1$, and $u$ by Figure 5ix. Then, we color the remaining vertex $u_1$.

(xii) We restrict $u$ by one color to color $u'_1$, $u'_2$, and $u'_3$ last. We restrict $v'_1$ by one color to color $u$, $u_1$, and $u_2$ afterwards. We restrict $w$ by one color to color $w'_1$, $w'_2$, and $w'_3$ afterwards. We color the remaining vertices $w_1$, $w$, $v'_3$, $v'_2$, $v$, and $v'_1$ by Figure 5ix.

(xiii) We restrict $u$ by one color to color $u'_1$, $u'_2$, and $u'_3$ last. We color $w$ with $x \in L(w) \setminus L(w_1)$. Observe that $L(w_2) \subseteq L(w_1)$ so $x \notin L(w_1) \cup L(w_2)$. We color $w'_2$ and $w'_1$ in this order. The vertices $v_2$, $v_1$, $v$, $v'_3$, $v'_2$, $v'_1$, $u$, and $u_1$ are colorable by Figure 5x as $L(v_2) \subseteq L(v_1)$ and $L(v'_3) \subseteq L(v'_2)$. Now, we color the remaining vertices $w_1$ and $w_2$ in this order.

(xiv) We restrict $u$ by one color to color $u'_1$, $u'_2$, and $u'_3$ last. We restrict $w$ by one color to color $w'_1$, $w'_2$, and $w'_3$ afterwards. We color $w$ then the remaining vertices $v'_3$, $v'_2$, $v$, $v_1$, $v_2$, $v'_1$, $u$, and $u_1$ are colorable by Figure 5x as $L(v_2) \subseteq L(v_1)$ and $L(v'_3) \subseteq L(v'_2)$.

(xv) We restrict $u$ by one color to color $u'_1$, $u'_2$, and $u'_3$ last. We restrict $w$ by one color to color $w'_1$, $w'_2$, and $w'_3$ afterwards. We color $w$ then the remaining vertices $v'_4$, $v'_3$, $v$, $v_1$, $v'_1$, $v'_2$, $u$, and $u_1$ are colorable by Figure 5xi.

(xvi) We restrict $w$ by one color to color $w'_1$, $w'_2$, and $w'_3$ last. We color $u$ with $x \in L(u) \setminus L(u_1)$. Observe that $L(u_2) \subseteq L(u_1)$ so $x \notin L(u_1) \cup L(u_2)$. We color $u'_2$ and $u'_1$ in this order. We color $v'_2$, $v'_1$, $v$, $v_1$, $v'_3$, $v'_4$, $w$, and $w_1$ by Figure 5xi. Now, we color the remaining vertices $u_1$ and $u_2$ in this order.

(xvii) We color $w$ with $x \in L(w) \setminus L(w_1)$. Observe that $L(w_2) \subseteq L(w_1)$ so $x \notin L(w_1) \cup L(w_2)$. We color $v$ with $y \in L(v) \setminus L(v_1)$. Observe that $L(v_2) \subseteq L(v_1)$ so $y \notin L(v_1) \cup L(v_2)$. We color $w'_2$, $w'_1$, $v'_4$, and $v'_3$ in this order. We color $v'_1$, $v'_2$, $u$, $u_1$, $u_2$, $u'_1$, $u'_2$ by Figure 5viii as $L(u'_2) \subseteq L(u'_1)$, $L(u_2) \subseteq L(u_1)$, and $L(v'_1) \subseteq L(v'_2)$. Now, we color the remaining vertices $v_1$, $v_2$, $w_1$ and $w_2$ in this order.

(xviii) We restrict $w$ by one color to color $w'_1$, $w'_2$, and $w'_3$ last. We color $v$ with $x \in L(v) \setminus L(v_1)$. Observe that $L(v_2) \subseteq L(v_1)$ so $x \notin L(v_1) \cup L(v_2)$. We color $w$, $v'_4$, and $v'_3$ in this order. We color $v'_1$, $v'_2$, $u$, $u_1$, $u_2$, $u'_1$, $u'_2$ by Figure 5viii as $L(u'_2) \subseteq L(u'_1)$, $L(u_2) \subseteq L(u_1)$, and $L(v'_1) \subseteq L(v'_2)$. Now, we color the remaining vertices $v_1$ and $v_2$ in this order.

(xix) We restrict $w$ by one color to color $w'_1$, $w'_2$, and $w'_3$ last. We restrict $u$ by one color to color $u'_1$, $u'_2$, and $u'_3$ afterwards. We color $u$ and $w$ then the remaining vertices $v'_2$, $v'_1$, $v$, $v_1$, $v_2$, $v'_3$, $v'_4$ are colorable by Figure 5viii as $L(v'_2) \subseteq L(v'_1)$, $L(v_2) \subseteq L(v_1)$, and $L(v'_4) \subseteq L(v'_3)$.

□

### 2.3 Discharging rules

In this section, we will define a discharging procedure that contradicts the structural properties of $G$ (see Lemmas 11 to 13) showing that $G$ does not exist. We assign to each vertex $u$ the charge $\mu(u) = \frac{19}{2}d(u) - 21$ and to each face $f$ the charge $\mu(f) = d(f) - 21$. By Equation (1), the total sum of the charges is negative. We then apply the following discharging rules:

Let $u$ and $v$ be endvertices of a $m$-path where $u = (k, l, m)$ with $k + l + m \leq 7$ and $v = (m, n, p)$. Vertex $u$ gives charge to $v$ in the following cases:

**R0** If $m = 0$

- (i) and $v = (0, 5, 5)$, then $u$ gives $\frac{5}{2}$ to $v$.
- (ii) and $v = (0, 4, 5)$, then $u$ gives $\frac{3}{2}$ to $v$.
- (iii) and $v \in \{(0, 3, 5), (0, 4, 4)\}$, then $u$ gives $\frac{1}{2}$ to $v$.
- (iv) and $v = (0, 2, 5)$, then $u$ gives $\frac{1}{4}$ to $v$.

**R1** If $m = 1$

- (i) and $v \in \{(1, 3, 5), (1, 4, 4)\}$, then $u$ gives $\frac{3}{2}$ to $v$.
- (ii) and $v \in \{(1, 3, 4), (1, 2, 5)\}$, then $u$ gives $\frac{1}{2}$ to $v$.

**R2** If $m = 2$

- (i) and $v = (2, 2, 5)$, then $u$ gives $\frac{3}{4}$ to $v$.
- (ii) and $v \in \{(2, 3, 3), (2, 1, 5)\}$, then $u$ gives $\frac{1}{2}$ to $v$.
- (iii) and $v = (2, 2, 4)$, then $u$ gives $\frac{1}{4}$ to $v$.

**R3** Finally, every 3-vertex gives 1 to each 2-vertex on its incident paths.

### 2.4 Verifying that charges on each face and each vertex are non-negative

Let $\mu^*$ be the assigned charges after the discharging procedure. In what follows, we will prove that:

$$\forall u \in V(G), \mu^*(u) \geq 0 \text{ and } \forall f \in F(G), \mu^*(f) \geq 0.$$

First of all, since $G$ is connected (Lemma 9), has minimum degree at least 2 (Lemma 10), has girth at least 21, and the discharging rules do not interfere with charge on faces, every face $f$ verifies $\mu^*(f) = \mu(f) = d(f) - 21 \geq 0$.

Now, let $u$ be a vertex in $V(G)$. If $d(u) = 2$, then $u$ receives charge 1 from each endvertex of the path it lies on by **R3**; thus we get $\mu^*(u) = \mu(u) + 2 \cdot 1 = \frac{19}{2} \cdot 2 - 21 + 2 = 0$.

From now on, suppose that $d(u) = 3$ and let $u = (k, l, m)$. Recall that $\mu(u) = \frac{19}{2} \cdot 3 - 21 = \frac{15}{2}$:

**Case 1:** Suppose that $k + l + m \geq 8$.
First, observe that $u$ only gives away charges by **R3**. More precisely, $u$ gives a total of $k + l + m$ to 2-vertices. Since there are no $6^+$-paths, $(1^+, 4^+, 5^+)$, $(2^+, 3^+, 4^+)$, or $(3^+, 3^+, 3^+)$ due to Lemma 11, then the only possible values for $k$, $l$, and $m$ are as follows:

- If $u$ is a $(5, 5, 0)$, $(5, 4, 0)$, $(5, 3, 0)$ or $(4, 4, 0)$, then $u$ cannot be adjacent to a vertex $v = (m, n, p)$ with $m + n + p \geq 8$ as $(430 - 024)$ is reducible by Lemma 12(i). As a result, $u$ receives charge $\frac{5}{2}$ (resp. $\frac{3}{2}$, $\frac{1}{2}$, or $\frac{1}{2}$) by **R0**(i) (resp. **R0**(ii), **R0**(iii), or **R0**(iii)) when it is a $(5, 5, 0)$ (resp. $(5, 4, 0)$, $(5, 3, 0)$, or $(4, 4, 0)$). To sum up, we have

$$\begin{aligned} \mu^*(u) &= \frac{15}{2} + \frac{5}{2} - 5 - 5 = 0 && \text{when } u = (5,5,0) \\ &= \frac{15}{2} + \frac{3}{2} - 5 - 4 = 0 && \text{when } u = (5,4,0) \\ &= \frac{15}{2} + \frac{1}{2} - 5 - 3 = 0 && \text{when } u = (5,3,0) \\ &= \frac{15}{2} + \frac{1}{2} - 4 - 4 = 0 && \text{when } u = (4,4,0) \end{aligned}$$

- If $u$ is a $(5, 3, 1)$, $(4, 4, 1)$, or $(4, 3, 1)$, then $u$ cannot share a 1-path with a vertex $v = (m, n, p)$ with $m + n + p \geq 8$ as $(431 - 114)$ is reducible by Lemma 12(iii). As a result, $u$ receives charge $\frac{3}{2}$ (resp. $\frac{3}{2}$, or $\frac{1}{2}$) by **R1**(i) (resp. **R1**(i), or **R1**(ii)) when it is a $(5, 3, 1)$ (resp. $(4, 4, 1)$, or $(4, 3, 1)$). To sum up, we have

$$\begin{aligned} \mu^*(u) &= \frac{15}{2} + \frac{3}{2} - 5 - 3 - 1 = 0 && \text{when } u = (5,3,1) \\ &= \frac{15}{2} + \frac{3}{2} - 4 - 4 - 1 = 0 && \text{when } u = (4,4,1) \\ &= \frac{15}{2} + \frac{1}{2} - 4 - 3 - 1 = 0 && \text{when } u = (4,3,1) \end{aligned}$$

- If $u$ is a $(5, 2, 2)$ or $(4, 2, 2)$, then $u$ cannot share a 2-path with a vertex $v = (m, n, p)$ with $m+n+p \geq 8$ as $(422 - 223)$ and $(422 - 214)$ are reducible respectively by Lemma 12(iv) and Lemma 12(v). As a result, $u$ receives charge $\frac{3}{4}$ (resp. $\frac{1}{4}$) by **R2**(i) (resp. **R2**(iii)) when it is a $(5, 2, 2)$ (resp. $(4, 2, 2)$) twice (once from each incident 2-path). To sum up, we have

$$\begin{aligned} \mu^*(u) &= \frac{15}{2} + 2 \cdot \frac{3}{4} - 5 - 2 - 2 = 0 && \text{when } u = (5,2,2) \\ &= \frac{15}{2} + 2 \cdot \frac{1}{4} - 4 - 2 - 2 = 0 && \text{when } u = (4,2,2) \end{aligned}$$

- If $u$ is a $(3, 3, 2)$, then $u$ cannot share a 2-path with a vertex $v = (m, n, p)$ with $m+n+p \geq 8$ as $(412-233)$ and $(332 - 233)$ are reducible respectively by Lemma 12(vi) and (vii). As a result, $u$ receives charge $\frac{1}{2}$ by **R2**(ii). To sum up, we have
$$\mu^*(u) = \frac{15}{2} + \frac{1}{2} - 3 - 3 - 2 = 0$$

- If $u$ is a $(5, 2, 1)$, then $u$ cannot share a 2-path with a vertex $v = (l, i, j)$ with $l + i + j \geq 8$ and $u$ cannot share a 1-path with a vertex $w = (m, n, p)$ with $m + n + p \geq 8$ at the same time, as $(412 - 233)$ and $(421 - 132 - 214)$ are reducible respectively by Lemma 12(vi) and Lemma 13(xiv). As a result, $u$ receives at least charge $\frac{1}{2}$ by **R1**(ii) or **R2**(ii). To sum up, we have

$$\mu^*(u) \geq \frac{15}{2} + \frac{1}{2} - 5 - 2 - 1 = 0$$

**Case 2:** Suppose that $k+l+m \leq 7$ and that $u$ is a $(2^-, 5^-, 2^-)$.
First, observe that when $u$ is a $(2^-, 2^-, 2^-)$, it gives at most $\frac{5}{2}$ along every incident path except for the case of **R2**(i), when it shares a 2-path with a $(2,2,5)$. Indeed, by **R0**, $u$ gives at most $\frac{5}{2}$ to an adjacent 3-vertex. By **R1** and **R3**, $u$ gives 1 to the 2-vertex on the 1-path and at most $\frac{3}{2}$ to the other endvertex. By **R2**(ii), **R2**(iii), and **R3**, $u$ gives 2 to the 2-vertices on the 2-path and at most $\frac{1}{2}$ to the other endvertex. As a result, $u$, a $(2^-, 2^-, 2^-)$ that does not share a 2-path with a $(2,2,5)$, verifies

$$\mu^*(u) \geq \frac{15}{2} - 3 \cdot \frac{5}{2} = 0$$

In other words, for the following values of $k, l, m$, we only need to look at $2 \leq l \leq 5$. Moreover, when $l = 2$, we can assume w.l.o.g. that the other endvertex of the 2-path is a $(2,2,5)$ since $k$, $l$, and $m$ are interchangeable.

Let $v = (i, j, k)$ share the $k$-path with $u$ and let $w = (m, n, p)$ share the $m$-path with $u$ (see Figure 8). For each case, only **R3**, **Rk** and **Rm** apply, with the additional **R2**(i) when $l = 2$.

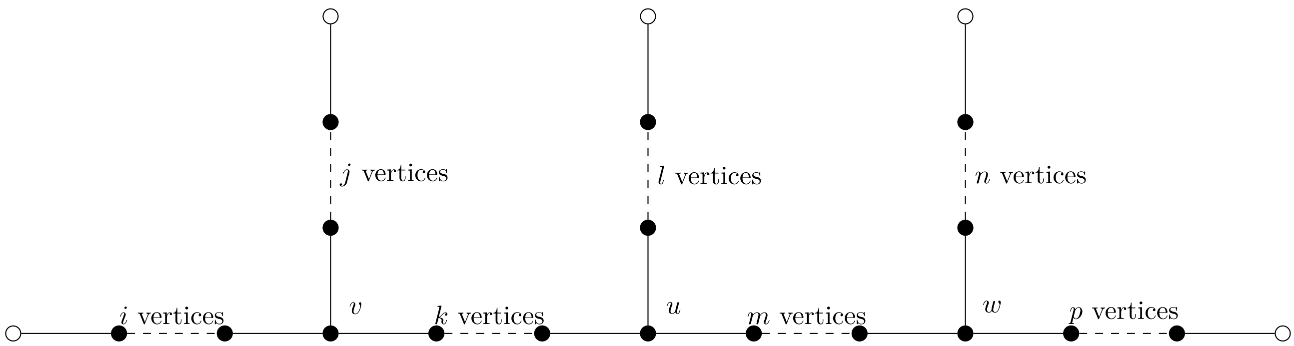



Figure 8

- If $u$ is a $(0, 5^-, 0)$, then we distinguish the two following cases:

  If $2 \leq l \leq 3$, then $u$ gives at most 3 along the $l$-path: either 3 to the 2-vertices in the case of a 3-path or 2 to the 2-vertices and $\frac{3}{4}$ to the other endvertex by **R2**(i). Since $(550 - 020 - 045)$ is reducible by Lemma 13(i), $u$ cannot give $\frac{5}{2}$ twice to $v$ and $w$ by **R0**. So at worst, we have

$$\mu^*(u) = \frac{15}{2} - 3 - \frac{5}{2} - \frac{3}{2} = \frac{1}{2}$$

  If $4 \leq l \leq 5$, then $u$ gives at most 5 to the 2-vertices along the $l$-path. Since $(440 - 040 - 024)$ is reducible by Lemma 13(ii), $u$ cannot give $\frac{3}{2}$ twice to $v$ and $w$ by **R0**. So at worst, we have

$$\mu^*(u) = \frac{15}{2} - 5 - \frac{3}{2} - \frac{1}{2} = \frac{1}{2}$$

- If $u$ is a $(0, 5^-, 1)$, then we distinguish the two following cases:

  If $2 \leq l \leq 3$, then $u$ gives at most 3 along the $l$-path: either 3 to the 2-vertices in the case of a 3-path or 2 to the 2-vertices and $\frac{3}{4}$ to the other endvertex by **R2**(i). Since $(550 - 021 - 134)$ and $(420 - 031 - 134)$ are reducible respectively by Lemma 13(iii) and (iv), $u$ cannot give $\frac{5}{2}$ twice to $v$ and $w$ by **R0** and **R1** ($1 + \frac{3}{2}$ in the case of **R1**(i)). So at worst, we have

$$\mu^*(u) = \frac{15}{2} - 3 - \frac{5}{2} - \frac{3}{2} = \frac{1}{2}$$

  If $4 \leq l \leq 5$, then $u$ gives at most 5 along the $l$-path.

  - If $w$ is a $(1, 3^+, 4^+)$, then $v$ cannot be a $(4^+, 2^+, 0)$ since $(420-031-134)$ is reducible by Lemma 13(iv). As a result, $u$ gives at most $\frac{3}{2}$ along the 1-path by **R1** and nothing to its adjacent 3-vertex by **R0**. So at worst, we have

$$\mu^*(u) = \frac{15}{2} - 5 - 1 - \frac{3}{2} = 0$$

  - If $w$ is not a $(1, 3^+, 4^+)$, then $u$ gives at most $\frac{1}{2}$ along the 1-path by **R1** and at most $\frac{1}{2}$ to its adjacent 3-vertex by **R0** since $(540 - 014)$ is reducible by Lemma 12(ii). So at worst, we have

$$\mu^*(u) = \frac{15}{2} - 5 - 1 - \frac{1}{2} - \frac{1}{2} = 1$$

- If $u$ is a $(0,5^-,2)$, then we distinguish the four following cases:

  If $l=2$, then $u$ gives $2+\frac{3}{4}$ along the 2-path by **R3** and **R2**(i). Since $(550-022-224)$ is reducible by Lemma 13(v), $v$ cannot be a $(5,5,0)$. As a result, $u$ gives at most $\frac{3}{2}$ to its adjacent 3-vertex by **R0** and $2+\frac{3}{4}$ along each 2-path by **R3** and **R2**. So at worst, we have

  $$\mu^*(u)=\frac{15}{2}-\frac{3}{2}-2\cdot\left(2+\frac{3}{4}\right)=\frac{1}{2}$$

  If $l=3$, then $u$ gives 3 along the $l$-path and 2 along the 2-path by **R3**.

  - If $v$ is a $(5,4^+,0)$, then $w$ cannot be a $(2,1^+,4^+)$ nor a $(2,3,3)$ as $(540-032-214)$ and $(540-032-233)$ are reducible respectively by Lemma 13(vi) and Lemma 13(vii). As a result, $u$ gives at most $\frac{5}{2}$ to $v$ by **R0** and nothing to $w$ by **R2**. So at worst, we have

    $$\mu^*(u)=\frac{15}{2}-3-2-\frac{5}{2}=0$$

  - If $v$ is not a $(5,4^+,0)$, then $u$ gives at most $\frac{1}{2}$ to $v$ by **R0** and at most $\frac{3}{4}$ to $w$ by **R2**. So at worst, we have

    $$\mu^*(u)=\frac{15}{2}-3-2-\frac{1}{2}-\frac{3}{4}=\frac{5}{4}$$

  If $l=4$, then $u$ gives 4 along the $l$-path and 2 along the 2-path by **R3**. Since $(430-024)$ is reducible by Lemma 12(i), $v$ cannot be a $(4^+,3^+,0)$. As a result, $u$ gives at most $\frac{1}{4}$ to $v$ by **R0** and at most $\frac{3}{4}$ to $w$ by **R2**. So at worst, we have

  $$\mu^*(u)=\frac{15}{2}-4-2-\frac{1}{4}-\frac{3}{4}=\frac{1}{2}$$

  If $l=5$, then $u$ gives 5 along the $l$-path and 2 along the 2-path by **R3**.

  - If $v$ is a $(4^+,2^+,0)$, then $w$ cannot be a $(2,1^+,4^+)$ nor a $(2,3,3)$ as $(420-042-214)$ and $(420-042-233)$ are reducible respectively by Lemma 13(viii) and Lemma 13(ix). Moreover, $v$ cannot be a $(4^+,3^+,0)$ since $(430-024)$ is reducible by Lemma 12(i). As a result, $u$ gives at most $\frac{1}{4}$ to $v$ by **R0** and nothing to $w$ by **R2**. So at worst, we have

    $$\mu^*(u)=\frac{15}{2}-5-2-\frac{1}{4}=\frac{1}{4}$$

  - If $v$ is not a $(4^+,2^+,0)$, then $v=(i,j,k)$ with $i+j+k\leq 7$. Thus, $u$ receives $\frac{1}{4}$ from $v$ by **R0**(iv). Moreover, $u$ gives nothing to $v$ by **R0** and at most $\frac{3}{4}$ to $w$ by **R2**. So at worst, we have

    $$\mu^*(u)=\frac{15}{2}-5-2+\frac{1}{4}-\frac{3}{4}=0$$

- If $u$ is a $(1,5^-,1)$, then we distinguish the three following cases:

  If $l=2$, then $u$ gives $2+\frac{3}{4}$ along the 2-path by **R3** and **R2**(i) and 1 to each 2-vertex on the 1-paths by **R3**. Since $(431-112-224)$ is reducible by Lemma 13(xii), $v$ cannot be a $(4^+,3^+,1)$. The same holds for $w$. As a result, $u$ gives at most $\frac{1}{2}$ twice to $v$ and $w$ by **R1**. So at worst, we have

  $$\mu^*(u)=\frac{15}{2}-2-\frac{3}{4}-1-1-2\cdot\frac{1}{2}=\frac{7}{4}$$

  If $l=3$, then $u$ gives 3 to the $l$-path and 1 to each 2-vertex on the 1-paths by **R3**. Since $(431-131-124)$ is reducible by Lemma 13(x), $v$ and $w$ cannot both be $(4^+,3^+,1)$s. As a result, $u$ cannot give $\frac{3}{2}$ twice by **R1**. So at worst, we have

  $$\mu^*(u)=\frac{15}{2}-3-1-1-\frac{3}{2}-\frac{1}{2}=\frac{1}{2}$$

  If $4\leq l\leq 5$, then $u$ gives at most 5 along the $l$-path, 1 to each 2-vertex on the 1-paths by **R3**. Since $(431-114)$ is reducible by Lemma 12(iii), $u$ cannot give more than $\frac{1}{2}$ to $v$ nor $w$ by **R1**. Moreover, since $(421-141-124)$ is also reducible by Lemma 13(xi), $u$ cannot give $\frac{1}{2}$ twice by **R1**. So at worst, we have

  $$\mu^*(u)=\frac{15}{2}-5-1-1-\frac{1}{2}=0$$

- If $u$ is a $(1,5^-,2)$, then $l \leq 4$ since $k+l+m \leq 7$. Thus, we distinguish the three following cases:

  If $l=2$, then $u$ gives $2+\frac{3}{4}$ along at least one of the 2-paths by **R3** and **R2**(i) and 1 to each 2-vertex on the 1-path and other 2-path by **R3**. Since $(431-112-224)$ is reducible by Lemma 13(xii), $v$ cannot be a $(4^+,3^+,1)$. As a result, $u$ gives at most $\frac{1}{2}$ to $v$ by **R1** and at most $\frac{3}{4}$ to $w$ by **R2**. So at worst, we have

  $$\mu^*(u) = \frac{15}{2} - 2 - \frac{3}{4} - 2 - 1 - \frac{1}{2} - \frac{3}{4} = \frac{1}{2}$$

  If $l=3$, then $u$ gives 3 along the $l$-path and 1 to each 2-vertex on the 1-path and 2-path by **R3**.

  – If $v$ is a $(4^+,2^+,1)$, then $w$ cannot be a $(2,1^+,4^+)$ nor a $(2,3,3)$ since $(421-132-233)$ and $(421-132-214)$ are reducible respectively by Lemma 13(xiii) and (xiv). As a result, $u$ gives at most $\frac{3}{2}$ to $v$ by **R1** and nothing to $w$ by **R2**. So at worst, we have

  $$\mu^*(u) = \frac{15}{2} - 3 - 2 - 1 - \frac{3}{2} = 0$$

  – If $v$ is not a $(4^+,2^+,1)$, then $u$ gives nothing to $v$ by **R1** and at most $\frac{3}{4}$ to $w$ by **R2**. So at worst, we have

  $$\mu^*(u) = \frac{15}{2} - 3 - 2 - 1 - \frac{3}{4} = \frac{3}{4}$$

  If $l=4$, then $u$ gives 4 along the $l$-path and 1 to each 2-vertex on the 1-path and 2-path by **R3**. Since $(431-114)$, $(422-214)$, and $(412-233)$ are reducible respectively by Lemma 12(iii), (v) and (vi), $v$ cannot be a $(4^+,3^+,1)$ and $w$ cannot be a $(2,2^+,4^+)$ nor a $(2,3,3)$. Moreover, $(412-132-214)$ is reducible by Lemma 13(xiv). As a result, $u$ can give at most $\frac{1}{2}$ once to either $v$ or $w$. So at worst, we have

  $$\mu^*(u) = \frac{15}{2} - 4 - 2 - 1 - \frac{1}{2} = 0$$

- If $u$ is a $(2,5^-,2)$, then $l \leq 3$, since $k+l+m \leq 7$. Thus, we distinguish the two following cases:

  If $l=2$, then $u$ gives $2+\frac{3}{4}$ along at least one of the 2-paths by **R3** and **R2**(i) and 1 to each 2-vertex on the 2-paths by **R3**. Since $(422-222-214)$ and $(332-222-224)$ are reducible respectively by Lemma 13(xv) and (xvi), $v$ cannot be a $(4^+,1^+,2)$ nor a $(3,3,2)$. The same holds for $w$. As a result, $u$ gives nothing to $v$ nor $w$ by **R2**. So at worst, we have

  $$\mu^*(u) = \frac{15}{2} - 2 - \frac{3}{4} - 2 - 2 = \frac{3}{4}$$

  If $l=3$, then $u$ gives 3 along the $l$-path and 1 to each 2-vertex on the 2-paths by **R3**.

  – If either $v$ or $w$ is a $(3,3,2)$, then the other cannot be a $(2,3,3)$ nor a $(2,1^+,4^+)$ as $(332-232-233)$ and $(332-232-214)$ are reducible respectively by Lemma 13(xvii) and (xviii). So, $u$ gives only $\frac{1}{2}$ once to either $v$ or $w$ by **R2**. So at worst, we have

  $$\mu^*(u) = \frac{15}{2} - 3 - 2 - 2 - \frac{1}{2} = 0$$

  – If neither $v$ nor $w$ is a $(3,3,2)$, then the remaining cases are as follows. Since $(422-223)$ is reducible by Lemma 12(iv), $v$ cannot be a $(4^+,2^+,2)$. The same holds for $w$. Moreover, since $(412-232-214)$ is reducible by Lemma 13(xix), they cannot both be $(4^+,1^+,2)$s. As a result, $u$ gives at most $\frac{1}{2}$ once to either $v$ or $w$ by **R2**. So at worst, we have

  $$\mu^*(u) = \frac{15}{2} - 3 - 2 - 2 - \frac{1}{2} = 0$$

**Case 3:** Suppose that $k+l+m \leq 7$ and that $u$ is a $(3^+,5^-,3^+)$.
Since $k+l+m \leq 7$, the only possibilities for $u$ are as follows:

- If $u$ is a $(3,0,3)$, then $u$ can only give charge by **R0** and **R3**. Since $(330-045)$ is reducible by Lemma 11(v), $u$ can give at most $\frac{1}{2}$ to another 3-vertex by **R0**(iii) or **R0**(iv). As a result,

  $$\mu^*(v) \geq \frac{15}{2} - \frac{1}{2} - 3 - 3 = 1$$

- If $u$ is a $(3,1,3)$, then $u$ can only give charge by **R1** and **R3**. Since $(431-133)$ is reducible by Lemma 11(vi), $u$ can give at most $\frac{1}{2}$ to another 3-vertex by **R1**(ii). As a result,

$$\mu^*(v) \geq \frac{15}{2} - \frac{1}{2} - 3 - 3 - 1 = 0$$

- If $u$ is a $(4,0,3)$, then $u$ can only give charge by **R0** and **R3**. Since $(430-024)$ is reducible by Lemma 12(i), $u$ actually does not give charge by **R0**. As a result,

$$\mu^*(v) \geq \frac{15}{2} - 4 - 3 = \frac{1}{2}$$

To conclude, we started with a charge assignment with a negative total sum, but after the discharging procedure, which preserved that sum, we end up with a non-negative one, which is a contradiction. In other words, there exists no counter-example $G$ to Theorem 4.

## 3 A non 4-colorable subcubic planar graph of girth 11

In [18], Dvořák, Škrekovski, and Tancer presented a non 4-colorable, planar, and subcubic graph with girth at least 9. The main building block of that graph relies upon an interesting property of 4-colorings on path of length 5. Using the same property we managed to build a non 4-colorable planar subcubic graph of girth 11.

**Lemma 14.** *Let $H$ be a subcubic graph of girth at least 11 and $\phi$ a 4-coloring of $H$. Let $u_1u_2u_3u_4u_5u_6$ be a path of length 5 in $H$, if $\phi(u_1) = \phi(u_6)$, then $\phi(u_2) = \phi(u_5)$.*

*Proof.* Since $H$ has girth at least 11, all considered vertices are distinct. Suppose by contradiction that $\phi(u_1) = \phi(u_6)$ but $\phi(u_2) \neq \phi(u_5)$. W.l.o.g. we set $\phi(u_1) = \phi(u_6) = a$, $\phi(u_2) = b$, and $\phi(u_5) = c$. Since $u_3$ sees $u_1$, $u_2$, and $u_5$, colored respectively $a$, $b$, and $c$, it must be colored $d$. Finally, $u_4$ sees $u_2$, $u_3$, $u_5$, and $u_6$, colored respectively by $b$, $d$, $c$, and $a$. Thus, $u_4$ is non-colorable, which is a contradiction since $\phi$ is a 4-coloring of $H$. □

**Lemma 15.** *Let $H$ be a subcubic graph of girth 11 and $\phi$ a 4-coloring of $H$. Let $u_1u_2u_3u_4u_5u_6$, $u_3u'_1u'_2u'_3u'_4v_1$, $u_4u''_1u''_2u''_3u''_4v_1$ be paths of length 5 in $H$. Let $v_0 \notin \{u'_4, u''_4\}$ be adjacent to $v_1$. If $\phi(u_1) = \phi(u_6) = \phi(v_0)$, then $\phi(u_2) = \phi(u_5) = \phi(v_1)$.*

*Proof.* Since $H$ has girth 11, all considered vertices are distinct. We assume w.l.o.g. that $\phi(u_1) = \phi(u_6) = \phi(v_0) = a$. By Lemma 14, since $\phi(u_1) = \phi(u_6)$, we must have $\phi(u_2) = \phi(u_5)$. W.l.o.g. we set $\phi(u_2) = \phi(u_5) = b$. As a result, we have $\{\phi(u_3), \phi(u_4)\} = \{c, d\}$. We assume w.l.o.g. that $\phi(u_3) = c$ and $\phi(u_4) = d$. Now, suppose by contradiction that $\phi(v_1) = c$. By Lemma 14, since $\phi(u_3) = \phi(v_1)$, we must have $\phi(u'_1) = \phi(u'_4) = a$. However, this is impossible since $u'_4$ sees $v_0$ which is colored $a$. By symmetry, the same argument holds when $\phi(v_1) = d$. Finally, since $v_1$ also sees $v_0$, thus $\phi(v_1) \notin \{a, c, d\}$, and so $\phi(v_1) = b = \phi(u_2) = \phi(u_5)$. □

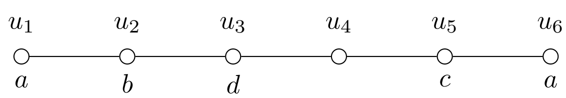



Figure 9: A non-valid coloring of $H$ in Lemma 14.

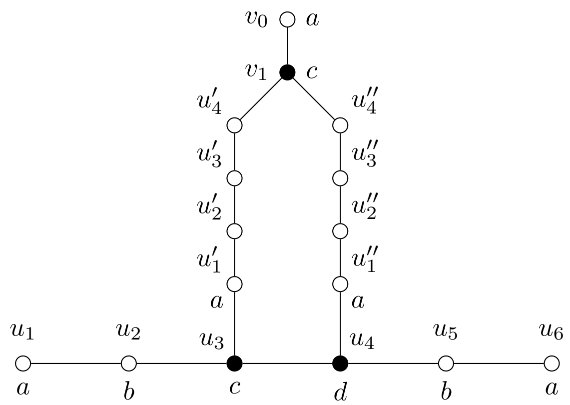



Figure 10: A non-valid coloring of $H$ in Lemma 15.

**Lemma 16.** *The graph $G_{\neq}(u, v)$ in Figure 11i has the following properties:*

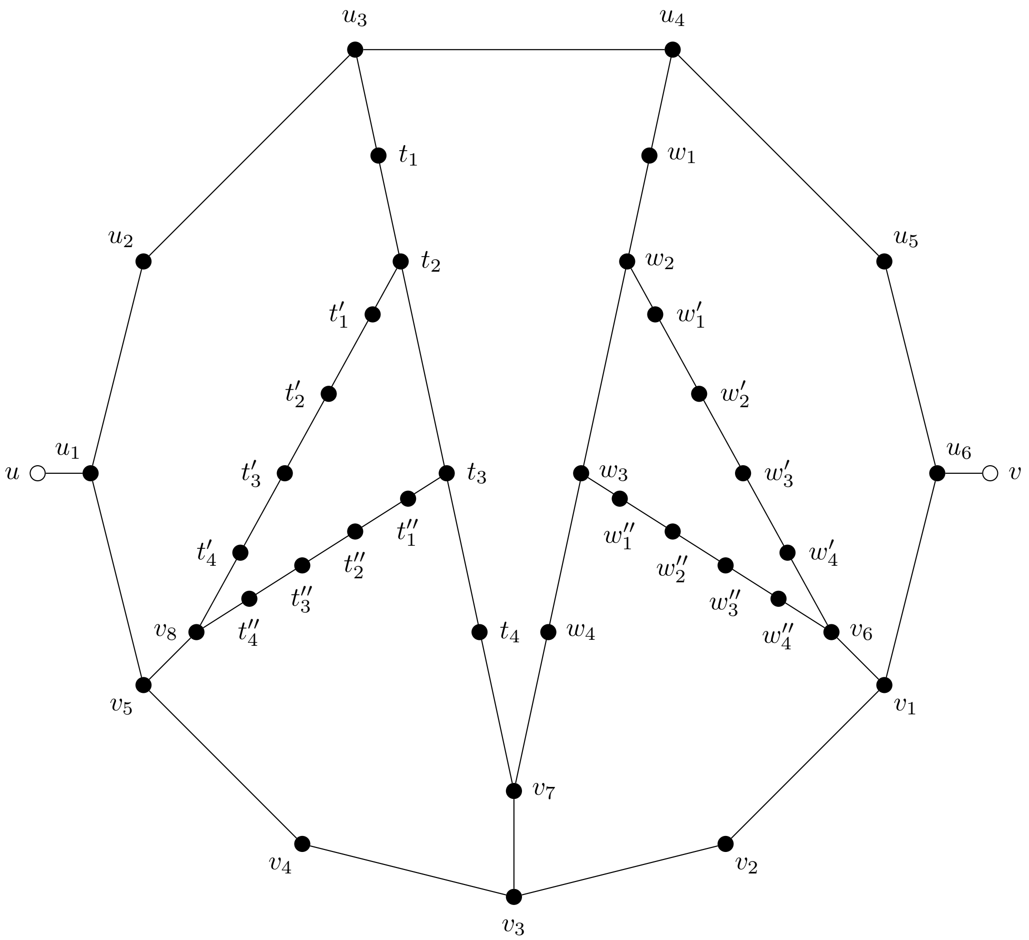
- *$G_{\neq}(u,v)$ is planar and subcubic.*
- *$G_{\neq}(u,v)$ has girth 11.*
- *The distance in $G_{\neq}(u,v)$ between $u$ and $v$ is 7.*
- *Every 4-coloring $\phi$ of $G_{\neq}(u,v)$ satisfies $\phi(u) \neq \phi(v)$.*

*Proof.* One can verify that $G_{\neq}(u,v)$ is planar, subcubic, has girth 11, and that the distance between $u$ and $v$ is 7 thanks to Figure 11i. It remains to prove that $\phi(u) \neq \phi(v)$ for every 4-coloring $\phi$ of $G_{\neq}(u,v)$.

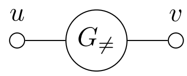
Suppose by contradiction that there exists a 4-coloring $\phi$ such that $\phi(u) = \phi(v) = a$. We can assume w.l.o.g. that $\phi(u_1) = b$, $\phi(u_2) = c$, and $\phi(v_5) = d$. Since $u_6$ sees $v$ which is colored $a$, we distinguish the following cases based on $\phi(u_6)$:

- If $\phi(u_6) = b$, then $\phi(u_5) = \phi(u_2) = c$ by Lemma 14 as $\phi(u_6) = \phi(u_2)$. As a result, $\phi(v_1) = d$. Since $v_2$ and $v_4$ both see $b$ and $d$, we have $\{\phi(v_2), \phi(v_4)\} = \{a, c\}$. Now, $v_3$ sees $\{\phi(v_1), \phi(v_2), \phi(v_4), \phi(v_5)\} = \{d, a, c\}$, so $\phi(v_3) = b$. Finally, $v_7$ sees $\{\phi(v_2), \phi(v_3), \phi(v_4)\} = \{a, b, d\}$, hence $\phi(v_7) = d$. However, this is impossible since $\phi(u_1) = \phi(u_6) = \phi(v_3) = b$, thus $\phi(u_2) = \phi(u_5) = \phi(v_7) = c$ by Lemma 15.
- If $\phi(u_6) = c$, then we have the two following cases:
  - If $\phi(v_1) = b$, then $\phi(v_2) = \phi(v_5) = d$ by Lemma 14 as $\phi(v_1) = \phi(u_1)$. As a result, $\phi(u_5) = d$ and $\phi(v_6) = a$. Since $v_3$ and $v_4$ both see $b$ and $d$, we have $\{\phi(v_3), \phi(v_4)\} = \{a, c\}$. Now, $v_7$ sees $\{\phi(v_2), \phi(v_3), \phi(v_4)\} = \{d, a, c\}$, so $\phi(v_7) = b$. Since $u_3$ sees $b$, $c$, and $d$, $\phi(u_3) = a$ and consequently, $\phi(u_4) = b$ and $\phi(w_1) = c$. However, this is impossible since $\phi(u_4) = \phi(v_7) = \phi(v_1) = b$, thus $\phi(w_1) = \phi(w_4) = \phi(v_6) = a$ by Lemma 15.
  - If $\phi(v_1) = d$, then $\phi(u_5) = b$. All three vertices $v_2$, $v_3$, and $v_4$ see $d$, so $\{\phi(v_2), \phi(v_3), \phi(v_4),\} = \{a, b, c\}$. As a result, $\phi(v_7) = d$. Both $u_3$ and $u_4$ see $b$ and $c$, so $\{\phi(u_3), \phi(u_4)\} = \{a, d\}$. Since $w_1$ sees $\{\phi(u_3), \phi(u_4), \phi(u_5)\} = \{a, d, b\}$, $\phi(w_1) = c$. Due to Lemma 15, we must have $\phi(u_4) = a$. Otherwise, by Lemma 15, $\phi(u_4) = d = \phi(v_7) = \phi(v_1)$ and $\phi(w_1) = \phi(w_4) = \phi(v_6) = c$ which is impossible since $v_6$ sees $u_6$ colored $c$. Thus, $\phi(u_3) = d$ and $\phi(t_1) = b$. However, this is also impossible since $\phi(u_3) = \phi(v_7) = \phi(v_5) = d$, thus $\phi(t_1) = \phi(t_4) = \phi(v_8) = b$ by Lemma 15 and $v_8$ sees $u_1$ colored $b$.
- If $\phi(u_6) = d$, then $\phi(v_1) = \phi(v_4)$ by Lemma 14 as $\phi(u_6) = \phi(v_5)$. Since $v_4$ sees $b$ and $d$ and $v_1$ sees $a$ and $d$, $\phi(v_4) = \phi(v_1) = c$. As a result, $\phi(u_5) = b$ and $\phi(v_8) = a$. Both $v_2$ and $v_3$ see $c$ and $d$, so $\{\phi(v_2), \phi(v_3)\} = \{a, b\}$. Now, $v_7$ sees $\{\phi(v_2), \phi(v_3), \phi(v_4)\} = \{a, b, c\}$, so $\phi(v_7) = d$. Since $u_4$ sees $d$, $b$, and $c$, $\phi(u_4) = a$ and consequently, $\phi(u_3) = d$ and $\phi(t_1) = b$. However, this is impossible since $\phi(u_3) = \phi(v_7) = \phi(v_5) = d$, thus $\phi(t_1) = \phi(t_4) = \phi(v_8) = a$ by Lemma 15.

□




(i) The gadget $G_{\neq}(u,v)$ in Lemma 16.

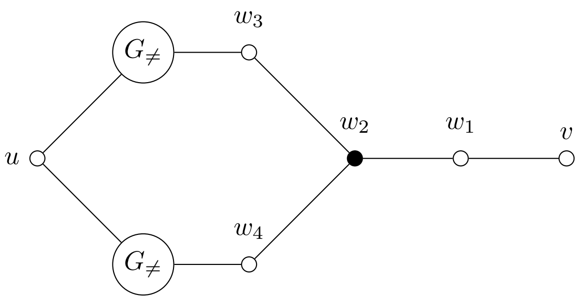



(ii) Simplified drawing of $G_{\neq}(u,v)$.

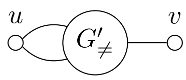
Figure 11

**Lemma 17.** *The graph $G'_{\neq}(u,v)$ in Figure 11i has the following properties:*

- *$G'_{\neq}(u,v)$ is planar and subcubic.*
- *$G'_{\neq}(u,v)$ has girth 11.*
- *The distance in $G'_{\neq}(u,v)$ between $u$ and $v$ is 10.*
- *Every 4-coloring $\phi$ of $G'_{\neq}(u,v)$ satisfies $\phi(u) \neq \phi(v)$.*




(i) The gadget $G'_{\neq}(u,v)$ in Lemma 17.




(ii) Simplified drawing of $G'_{\neq}(u,v)$.

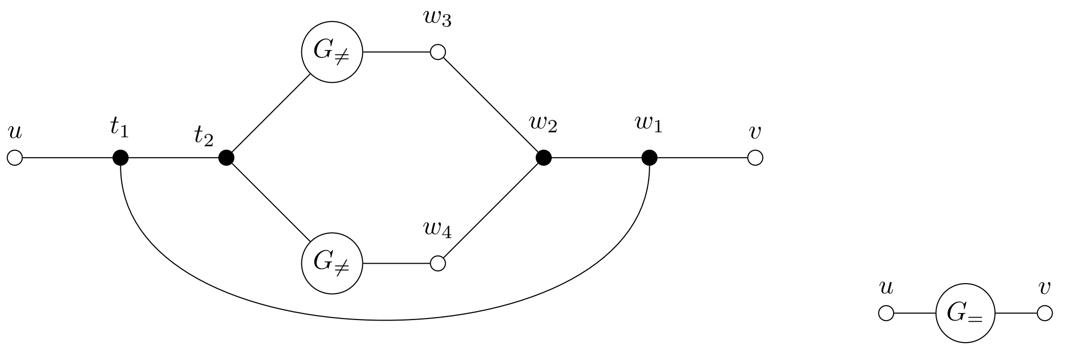
Figure 12

*Proof.* One can verify that $G'_{\neq}(u,v)$ is planar, subcubic, has girth 11, and that the distance between $u$ and $v$ is 10 thanks to Figure 12i and Lemma 16. It remains to prove that $\phi(u) \neq \phi(v)$ for every 4-coloring $\phi$ of $G'_{\neq}(u,v)$. Suppose by contradiction that there exists a 4-coloring $\phi$ of $G'_{\neq}(u,v)$ such that $\phi(u) = \phi(v)$, say $\phi(u) = a$. We only need to observe that $w_3$ and $w_4$ cannot be colored $a$ thanks to $G_{\neq}(u,v)$ and $w_1$ and $w_2$ cannot be colored $a$

since they see $v$. This is a contradiction as we have four vertices at distance two pairwise but only three colors left. $\square$

**Lemma 18.** *The graph $G_=(u,v)$ in Figure 13i has the following properties:*

- *$G_=(u,v)$ is planar and subcubic.*
- *$G_=(u,v)$ has girth 11.*
- *The distance in $G_=(u,v)$ between $u$ and $v$ is 3.*
- *Every 4-coloring $\phi$ of $G_=(u,v)$ satisfies $\phi(u)=\phi(v)$.*




(i) The gadget $G_=(u,v)$ in Lemma 18.

(ii) Simplified drawing of $G_=(u,v)$.

Figure 13

*Proof.* One can verify that $G_=(u,v)$ is planar, subcubic, has girth 11, and that the distance between $u$ and $v$ is 3 thanks to Figure 13i and Lemma 18. It remains to prove that $\phi(u)=\phi(v)$ for every 4-coloring $\phi$ of $G_=(u,v)$. Let $\phi$ be a 4-coloring of $G_=(u,v)$, we can assume w.l.o.g. that $\phi(u)=a$, $\phi(t_1)=b$, $\phi(t_2)=c$, and $\phi(w_1)=d$. Observe that $v$ sees $t_1$ and $w_1$ colored respectively $b$ and $d$. Moreover, due to Lemma 17, $\phi(v)\neq\phi(t_2)=c$ as $G_=(u,v)$ contains $G'_{\neq}(t_2,v)$. As a result, we must have $\phi(v)=a=\phi(u)$. $\square$

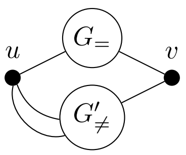



Figure 14: A non-4-colorable planar subcubic graph of girth 11.

As a direct consequence of Lemma 17 and Lemma 18, we get the following lemma.

**Lemma 19.** *The graph $G$ in Figure 14 is a planar subcubic graph of girth 11 with $\chi^2(G)\geq 5$.*

In [18], the authors also proved the NP-completeness of the problem of deciding if a planar subcubic graph of girth 9 is 4-colorable using a gadget that can reproduce colors at a far enough distance to preserve the girth condition. The same proof can be adapted directly to prove the NP-completeness of deciding if a planar subcubic graph of girth 11 is 4-colorable by using a concatenation of $G_=(u,v)$ to get a large enough distance.

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