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# Ricochet Robots with infinite horizontal board is Turing-complete<sup>\*</sup>

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**Abstract.** This paper investigates the Ricochet Robots game problem from a complexity standpoint. The problem consists of moving robots in a rectangular square-tiled board, from initial tiles to reach specific target tiles. A robot can only move vertically or horizontally and when it starts to move in a given direction, the robot follows this direction, until being blocked by a wall or another robot. This paper proves that the corresponding decision problem to Ricochet Robots is Turing-complete for endless board game and an infinite number of robots. A reduction from a universal Turing machine to Ricochet Robots is exhibited.

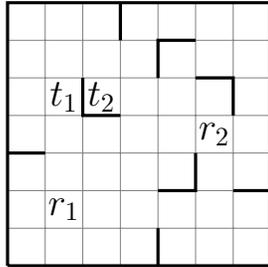
## 1 Introduction

“Ricochet Robots” [1,4,5,6] is a puzzle board game designed by A. Randolph, in which a player moves pieces (robots) in a rectangular square-tiled board from an initial position to a given set of selected locations, with the fewest possible moves. The game-board is a rectangular square-tiled board that contains walls placed on the edges of some tiles. Robots can move horizontally or vertically on the board. A robot moves in a direction until being blocked by a wall or another robot. Each step consists of selecting both a robot and a direction that the robot will follow. To solve the puzzle, the player must reach a configuration where all target tiles are covered by robots of corresponding colors. It is often necessary to move a robots that serves as guide to stop the movement of another to an appropriate tile (see Figure 1 for an example). Several robots cannot move at the same time. In the original game, the board is a square ( $16 \times 16$  tiles) which contains four different colored robots and one colored target tile. The color of the target tile corresponds to the color of one of the robots. The player can move all robots and must reach the target tile with the robot of the same color.

Ricochet Robots game can be categorized as a sliding game like the PushPush game studied by Demaine et al. [2] or the Atomix game studied by Holzer and Schwoon [8] and Huffner et al. [10]. Icking et al. [11,12] considered the exploration problem of a grid polygon with or without obstacles inside it. Engels and

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In this grid,  $r_1$  cannot be placed on the target tile  $t_1$  without the help of  $r_2$ . If the robot  $r_2$  reaches the target tile  $t_2$  with the two movements  $\uparrow$  then  $\leftarrow$  and ceases to move, then  $r_1$  will never reach  $t_1$ . A possible solution for this instance is to execute the following moves:  $r_2$ :  $\leftarrow$  ;  $r_1$ :  $\uparrow, \leftarrow, \downarrow, \rightarrow$  ;  $r_2$ :  $\uparrow, \rightarrow, \downarrow$ .

**Fig. 1.** Example of an instance of Ricochet Robots. To solve it,  $r_1$  must be placed on the target tile  $t_1$  while  $r_2$  must be on  $t_2$  at the same time.

Kamphans [4] studied the solvability of Ricochet Robots with  $n$  uncolored robots and one target tile, and proved that this problem is NP-complete. Hesterberg and Kopinsky [7] studied the parameterized complexity of Ricochet Robots and Atomix. Gebser et al. [5,6] used Ricochet Robots game as a benchmark for answer set programming while Butko et al. [1] proposed to study how humans try to solve Ricochet Robots. The same authors reused the result of Holzer and Schwon on Atomix to show that Ricochet Robots is also PSPACE-complete [3]. Another proof of the PSPACE-completeness of Ricochet Robots was given independently by Masseport et al. [13].

The contribution of this article is the construction presented in Section 4: any Turing machine can be simulated by a Ricochet Robots game with an endless board and an infinite number of robots. We can construct an instance of Ricochet Robots whose solvability depends on whether the Turing machine halts or not. This construction establishes the following theorem:

**Theorem 1.** *Ricochet Robots is Turing-complete.*

Section 2 provides background information on this work and several gadgets with specific properties are presented in Section 3. These gadgets are used in Section 4 to prove that Ricochet Robots with an endless board and an infinite number of robots is Turing-complete.

## 2 Preliminaries

### 2.1 Ricochet Robots

The original game implies four different colored robots and one colored target tile. The decision problem of GENERALIZED REACHABILITY (GR) is defined as a generalization of REACHABILITY PROBLEM introduced by Engels and Kamphans [4].

A *board game* is a rectangular square-tiled board (all the tiles are the same size and are vertically and horizontally aligned) that contains horizontal and

vertical walls between some tiles. In the following, we define a board  $B$  as a set of walls. Some tiles of the board game, called *target tiles* are colored. A *robot* is a colored token on a tile. A tile cannot contain more than one robot at the same time. An instance of Ricochet Robots  $I = (G, R, T)$  is constituted by a board game  $B$ , a set of robots  $R$  and a set of target tiles  $T$ . A configuration is *winning* if each colored target tile is covered by a robot of the same color. To reach a winning configuration, at each step, the player can move a robot vertically or horizontally on the game board. When a robot starts to move, it follows this direction, until it hits an obstacle (*i.e.* another robot or a wall). Target tiles do not stop movement. The GENERALIZED REACHABILITY (GR) problem is defined as follows:

GENERALIZED REACHABILITY (GR)

**Input:** An instance of Ricochet Robots  $I = (B, R, T)$

**Question:** Is there a reachable winning configuration?

The result of Turing-completeness presented in Section 4 uses a single color for target tiles and robots. For simplicity, the color is not specified. The construction depicted in the following creates an instance of Ricochet Robots of infinite size in the horizontal dimension. In order to show that the reduction is a many-one reduction, the instance resulting from it must be expressed in a finite way, otherwise the reduction would not consist in a computable function because the production of the instance would not terminate. Hence, in the following an infinite-size board is represented in a finite way by using something close to regular expressions to repeat some patterns indefinitely. Let  $I_1 = (B_1, R_1, T_1)$  and  $I_2 = (B_2, R_2, T_2)$  two instances of GR such that the horizontal size of  $B_1$  is equal to  $x_1$ . The instance  $I_1 \circ I_2$  is created by juxtaposing  $I_2$  after  $I_1$ . That is,  $I_1 \circ I_2 = (B_1 \cup B'_2, R_1 \cup R'_2, T_1 \cup T'_2)$  where  $(B'_2, R'_2, T'_2)$  is obtained by a horizontal translation of value  $x_1$  applied to  $I_2$  (*i.e.* the walls of the board, robots and target tiles have been shifted to the right). The instance  $I^*$  corresponds to an infinite number of juxtapositions of the instance  $I$ , in other words:  $I^* = I \circ I \circ \dots$

## 2.2 Turing machine

A Turing machine defines an abstract machine which manipulates symbols on a tape. This tape is divided into “cells” and the machine has a “head” positioned over a cell. This head reads and writes symbols on the tape. The cell on which the head is positioned is denoted as the *current cell*. A Turing machine has many definitions. Hopcroft [9] describes a Turing machine  $M$  as a 7-tuple  $(Q, \Sigma, \Gamma, \delta, q_0, Y_0, F)$  where:

- $Q$  is a finite, non-empty set of states,
- $\Sigma$  is a finite set of input symbols,
- $\Gamma$  is the complete set of tape symbols,  $\Sigma$  is always a subset of  $\Gamma$ ,
- $\delta(q, X) \rightarrow (p, Y, D)$  is a transition function which given a state  $q$  and a tape symbol  $X$  returns a triple containing, a next state  $p$ , a symbol  $Y$  written in

the cell being scanned and a direction  $D$ , either  $L$  (“left”) or  $R$  (“right”), in which the head moves.

- $q_0 \in Q$  is the initial state of  $M$ ,
- $Y_0 \in \Gamma \setminus \Sigma$  is a blank symbol, and
- $F \subseteq Q$  is a set of final or accepting states.

At each step, a Turing machine has a current state  $q$ . The head starts by reading the symbol  $X$  on the current cell. Then, according to the value returned by the transition function  $\delta$  with input  $(q, X)$ , the current state changes to  $p \in Q$ , the head writes a new symbol  $Y \in \Gamma$  in the current cell and moves to the left or the right cell on the tape.

A universal Turing machine is a Turing machine that can simulate any arbitrary Turing machine on arbitrary input. The universal machine essentially achieves this by reading both the description of the machine to be simulated and the input to that machine from its own tape.

### 3 Description of constructions

#### 3.1 Wires Representation

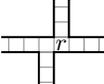
To simplify both our scheme and reader understanding’s, gadgets are represented by rectangles that are connected by “wires”. A wire corresponds to a line (possibly bent) of free spaces surrounded by walls on both sides. Two wires can cross each other to preserve planarity of the grid (see Table 1). Note that a robot passing through an intersection cannot change its way. In other words, a robot that gets in vertically cannot reach a horizontal output and vice versa. Wires, bent wires and crossed wires guarantee the planarity of the construction.

A helping robot wire is a particular connection of wires: if a robot is on the tile tagged by  $r$  (see Table 1), then it can “help” another robot coming from the left to go down. Notice the robot on the tile  $r$  cannot go down. In this paper, considering  $r_1$  a helping robot and  $r_2$  a robot needing help, “ $r_1$  intercepts  $r_2$  to reach the correct output (*i.e.* to go down)” means that  $r_1$  goes to the position to help  $r_2$  reach the correct output (*i.e.* the down output). Likewise, considering two gadgets  $G_1$  and  $G_2$  and a wire  $w$ , “ $G_1$  intersects  $w$  to  $G_2$ ” means that an output of  $G_1$  and an input of  $G_2$  are connected to  $w$  in such a way that a robot coming from  $G_1$  can help a robot coming from  $w$  to reach  $G_2$  by stopping it.

#### 3.2 Basic Gadgets

This section presents some basic gadgets and their properties. These gadgets are used in the next section for construction of more sophisticated gadgets. A *k-router gadget* (Figure 2) is a gadget with  $k'$  inputs and  $k$  outputs ( $k'$  and  $k > 0$ ) that has the following property:

**Table 1.** Wires representations.

	Representation	Corresponding grid
wire		
bent wire		
crossed wires		
helping robot wire		

**Property 1 (Router Property)** *When a robot reaches a router gadget (from an input or an output), it can reach any output.*

A  $k$ -synchronizer gadget is a gadget with  $k$  inputs and  $k$  outputs (with  $k > 0$ ). A 2-synchronizer gadget (resp. a 3-synchronizer gadget) is depicted in Figure 2 (resp. Figure 3). A synchronizer gadget has the following property:

**Property 2 (Synchronize Property)** *Let  $S$  be a  $k$ -synchronizer gadget and suppose that  $k'$  robots enter  $S$ . If  $k' < k$ , then no robot can reach an output. If  $k' = k$ , then each robot can reach a distinct output (i.e. two robots cannot reach the same output).*

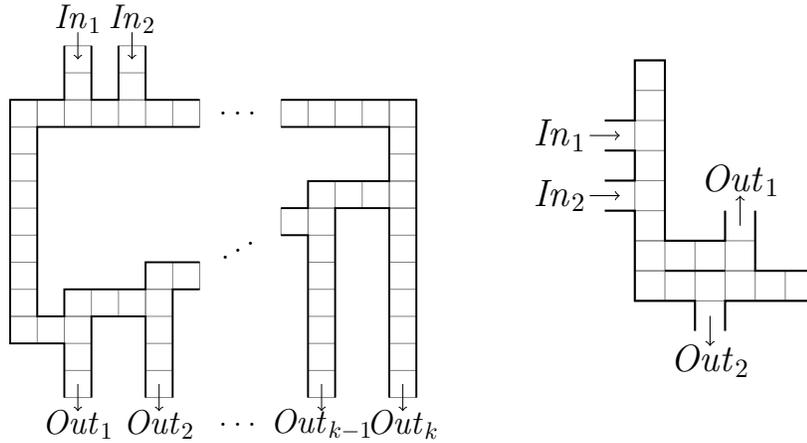
The case where the number of robots in a  $k$ -synchronizer gadget is strictly greater than  $k$  is not analyzed because it can not happen in the proposed construction. The next property is defined in order to clarify the construction:

**Property 3 ( $k$ -No-Return Property)** *If there are at most  $k > 0$  robots in a gadget that has a  $k$ -No-Return Property and at least one of them has reached an output, then none of them can go back to any input.*

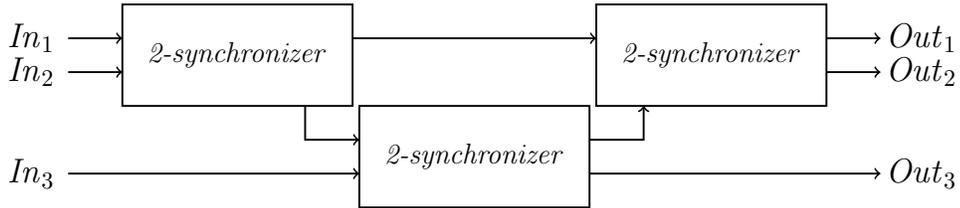
Obviously, a  $k$ -No-Return Property implies a  $(k - 1)$ -No-Return Property. Note that all gadgets defined in this paper have a 1-No-Return Property. A  $k$ -synchronizer gadget has a  $k$ -No-Return Property.

## 4 Turing-completeness

This section is devoted to proving the Turing-completeness of Ricochet Robots game. In order to show this, a many-one reduction from a universal Turing machine to Ricochet Robots is exhibited. Let  $M = (Q, \Sigma, \Gamma, \delta, q_0, Y_0, F)$  be an arbitrary Turing machine with  $m$  states ( $Q = \{q_0, \dots, q_{m-1}\}$ ) and  $n$  symbols ( $\Gamma = \{Y_0, \dots, Y_{n-1}\}$ ). The following construction is divided in two main gadgets:



**Fig. 2.** **Left:** a  $k$ -router gadget with two inputs. In this gadget, a robot can reach any output (while traveling through counterclockwise). **Right:** A 2-synchronizer gadget. Let  $r_1$  (resp.  $r_2$ ) be a robot that reaches the input  $In_1$  (resp.  $In_2$ ). In order to get  $r_1$  and  $r_2$  out of the gadget, one of them needs to reach the output  $Out_1$  and the other the output  $Out_2$  (they cannot reach the same one). Possible moves to cross this gadget after  $r_1$  and  $r_2$  have reached the input:  $r_2$ :  $\downarrow$ ;  $r_1$ :  $\downarrow, \rightarrow, \downarrow$ ;  $r_2$ :  $\rightarrow, \downarrow$ ;  $r_1$ :  $\uparrow$ .



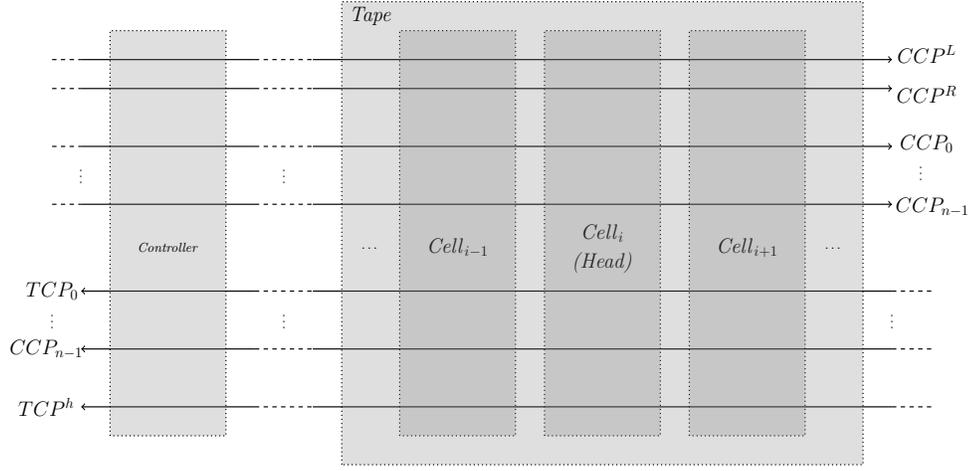
**Fig. 3.** A 3-synchronizer gadget. See Section 3.1 for details on the representation system.

- **Tape gadget** used to encode the symbols written on the tape of  $M$ .
- **Controller gadget** used to encode the current state of  $M$ .

The next subsection describes how the tape gadget and the controller gadget communicate. The Section 4.2 presents the tape gadget and how the read and write operations are simulated. Further, the execution of the transition function by the controller gadget is defined in Section 4.3. Finally, Section 4.4 shows that Ricochet Robots game is Turing-complete.

#### 4.1 Communication system

A *communication pipe* is an endless corridor (*i.e.* a line of free spaces surrounded by walls on both sides) with some helping robots connections. The communication system between tape gadget and controller gadget is composed of several communication pipes. We distinguish between two types of pipes: the “TCP”



**Fig. 4.** The communication system. The arrows indicate the direction of communication of the lines and the value corresponds to the message. For example, the communication pipe  $TCP_0$  is used to send the message “ $Y_0$ ” from the tape to the controller. Note that a pipe cannot be used to communicate in both directions.

communication pipes are used by the tape gadget to send robots to the controller gadget while the “CCP” communication pipes are used by the controller gadget to send robots to the tape gadget.

**Construction 1 (Communication gadget)** Create three communication pipes  $TCP^h$ ,  $CCP^L$  and  $CCP^R$  and  $\forall Y_i \in \Gamma$  create two communication pipes  $TCP_i$  and  $CCP_i$ .

The  $TCP_i$  communication pipes are used by the tape gadget to transmit the symbol contained on the current cell whereas the  $CCP_i$  communication pipes are used by the controller gadget to transmit the new value of the cell (*i.e.* a pipe cannot be used to communicate in both directions). Precisely, the tape gadget (resp. controller gadget) sends a robot through  $TCP_i$  (resp.  $CCP_i$ ) if the value (resp. the new value) on the current cell is  $Y_i$ . The controller indicates if the head has to move to the left or to the right by sending a robot through  $CCP^L$  or  $CCP^R$ , respectively. The communication pipe  $TCP^h$  is used to indicate the end of the reading action to the controller. See Figure 4 for the description of the communication system.

## 4.2 Head and Tape

This subsection introduces the gadget used to simulate both the head and the tape. The *tape gadget* consists of several *cell gadgets* defined below.

**Construction 2 (Tape gadget)** Given a communication gadget produced by Construction 1, construct a tape gadget as follows. Let  $c_i$  be the  $i^{\text{th}}$  cell of the tape. For each cell  $c_i$ , construct a cell gadget  $Cell_i$  (see Figure 5) as follows:

- construct one  $n$ -router gadget  $TR_i$ ,
- construct one  $2(n+1)$ -router gadget  $TR'_i$ ,
- construct two 2-synchronizer gadgets  $Left_i$  and  $Right_i$ , and
- for each  $Y_j \in \Gamma$ , construct two 2-synchronizer gadgets,  $Y_j^i$  and  $Write(Y_j^i)$ .

Inner gadgets of  $Cell_i$  are connected in the following way:

- intersect  $CCP^L$  with  $TR'_i$  to  $Left_i$ ,
- intersect  $CCP^R$  with  $TR'_i$  to  $Right_i$ ,
- for each  $Y_j \in \Gamma$ , connect an output of  $Write(Y_j^i)$  and an output of  $TR_i$  with the inputs of  $Y_j^i$  and connect the outputs of  $Y_j^i$  with the communication pipes  $TCP_j$  and  $TCP^h$ , and
- for each  $Y_j \in \Gamma$  intersect the communication pipe  $CCP_j$  with  $TR'_i$  to  $Write(Y_j^i)$ , connect an output of  $TR'_i$  with an input of  $Write(Y_j^i)$  and connect the last output of  $Write(Y_j^i)$  with inputs of  $Left_i$  and  $Right_i$ .

Finally, connect  $Cell_i$  with  $Cell_{i-1}$  and  $Cell_{i+1}$ :

- connect the outputs of  $Left_i$  with the inputs of  $TR_{i-1}$  and  $TR'_{i-1}$ , and
- connect the outputs of  $Right_i$  with the inputs of  $TR_{i+1}$  and  $TR'_{i+1}$ .

In a Turing machine, the head must be able to carry out three actions (if the transition function allows it):

- read the symbol of the current cell,
- write a symbol in the current cell,
- move on the tape to the cell on the right or left.

In each cell gadget  $Cell_i$ , a robot  $r_i$  is used to encode the symbol written in the corresponding cell  $c_i$ . The robot  $r_i$  is located in the 2-synchronizer gadget  $Y_j^i$  if and only if the  $i^{\text{th}}$  cell of the tape contains the symbol  $Y_j$ . If the robot  $r_i$  is in the gadget  $Y_j^i$ ,  $Cell_i$  is said to contain  $Y_j$ . Two robots  $h_1$  and  $h_2$  are used to simulate the head of  $M$ . The robot  $h_2$  is located in the cell gadget  $Cell_i$  if and only if the head is over the  $i^{\text{th}}$  cell. In that case,  $Cell_i$  is the current cell gadget.

**Lemma 1.** *Let  $Cell_i$  be a gadget produced by Construction 2.*

1. *If robots  $r_i$  and  $h_1$  are located in gadgets  $Y_j^i$  and  $TR_i$ , then  $r_i$  and  $h_1$  reach the communication pipes  $TCP_j$  and  $TCP^h$ .*
2. *If robots  $r_i$  and  $h_1$  are coming respectively from communication pipe  $CCP_j$  and  $CCP^L$  (resp.  $CCP^R$ ) and  $h_2$  is located in  $TR'_i$ , then  $r_i$  reaches  $Y_j^i$ ,  $h_1$  reaches  $TR_{i-1}$  (resp.  $TR_{i+1}$ ) and  $h_2$  reaches  $TR'_{i-1}$  (resp.  $TR'_{i+1}$ ).*

*Proof.* 1. If robot  $h_1$  enters any gadget  $Y_k^i$  such that  $k \neq j$ , then by Property 2,  $h_1$  is stuck in the gadget  $Y_k^i$  and  $r_i$  in  $Y_j^i$ . Thus, suppose that  $h_1$  enters in  $Y_j^i$ . By Property 2, robots  $r_i$  and  $h_1$  reach the communication pipes  $TCP_j$  and  $TCP^h$ , respectively.

2. By Property 1,  $h_2$  can intercept successively both  $r_i$  and  $h_1$  to  $Write(Y_j^i)$  and  $Left_i$  (resp.  $Right_i$ ) respectively. If  $h_2$  enters any gadget  $Write(Y_k^i)$  such that

$k \neq j$ , then by Property 2,  $r_i, h_1$  and  $h_2$  are stuck in their respective gadget. Thus, suppose that  $h_2$  enters in  $Write(Y_j^i)$ . By Property 2,  $r_i$  and  $h_2$  reach  $Y_j^i$  and  $Left_i$  (resp.  $Right_i$ ), respectively. Then, by Property 2, robots  $h_1$  and  $h_2$  reach gadgets  $TR_{i-1}$  and  $TR'_{i-1}$  (resp.  $TR_{i+1}$  and  $TR'_{i+1}$ ) respectively.

When the head reaches the  $i^{th}$  cell, (*i.e.*  $h_1$  and  $h_2$  enter in  $TR_i$  and  $TR'_i$ , respectively) three operations are simulated in the following way.

- **Reading operation.** This operation is executed by sending  $r_i$  with the help of  $h_1$  to the controller gadget via  $TCP_j$  (Lemma 1(1)). Note that after this operation,  $h_1$  is sent to the controller via  $TCP^h$  (this action indicates the end of the reading operation and is required to execute the transition function). Red paths in Figure 5 depict an example of a reading operation.
- **Writing operation.** The controller gadget indicates the new value  $Y_k$  to  $Cell_i$  by sending back  $r_i$  via  $CCP_k$ . Thus, writing operation is simulated by intercepting  $r_i$  with  $h_2$  in  $CCP_k$  and sending it to  $Y_k^i$  by crossing  $Write(Y_k^i)$  (Lemma 1(2)). Blue paths in Figure 5 depict an example of such operation.
- **Moving operation.** The controller gadget indicates the direction in which the head has to move by sending back  $h_1$  via  $CCP^L$  or  $CCP^R$ . Thus, a moving operation is simulated by intercepting  $h_1$  with  $h_2$  to  $Left_i$  or  $Right_i$ . After the writing operation,  $h_2$  can join  $h_1$  in its gadget and they are sent to  $TR_{i-1}$  and  $TR'_{i-1}$ , if left, or  $TR_{i+1}$  and  $TR'_{i+1}$ , if right (Lemma 1(2)) Green and blue paths in Figure 5 depict an example of such operation.

### 4.3 Controller gadget

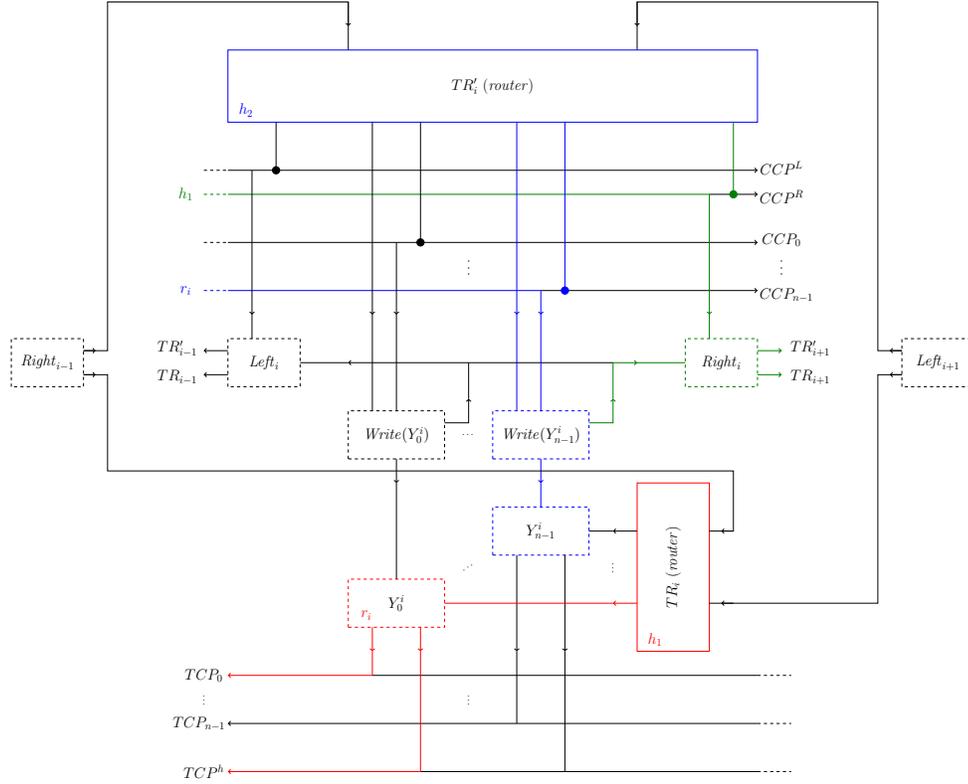
This subsection introduces the gadget used to simulate the transition function.

**Construction 3 (Controller gadget)** *Given a communication gadget produced by Construction 1, the controller gadget (see Figure 6 ) is constructed as follows. First, create a target tile  $t$ . Then, for each state  $q_i \in Q$ , construct a state gadget  $State_i$  as follows:*

- construct a  $(2n + 1)$ -router gadget  $CR_i$  and an  $n$ -router gadget  $CR'_i$ ,
- for each symbol  $Y_j \in \Gamma$ , construct a 3-synchronizer gadget  $\Delta_j^i$ .

*For each state  $q_i \in Q$  and for each symbol  $Y_j \in \Gamma$ , let  $(q_p, Y_\ell, D)$  be the values returned by  $\delta(q_i, Y_j)$  (with  $D \in \{L, R\}$ ). State gadgets are connected as follows:*

- connect an output of  $CR'_i$  with an input of  $\Delta_j^i$ ,
- connect an output of  $CR_i$  with an input of  $\Delta_j^i$ ,
- intersect  $TCP^h$  with  $CR_i$  to  $CR'_i$ ,
- intersect  $TCP_j$  with  $CR_i$  to  $\Delta_j^i$ ,
- connect the outputs of  $\Delta_j^i$  with the inputs of  $CR_p$ ,  $C'_{Y_\ell}$  and  $C'_D$  respectively,
- if  $\delta(q_i, Y_j)$  is a halting case for  $M$ , connect an output of  $\Delta_j^i$  to the target tile  $t$  and close the two other outputs with a wall.



**Fig. 5.** The cell gadget  $Cell_i$  representing the  $i^{th}$  cell of the tape of  $M$ . See Section 3.1 for details on the representation system. The output  $TR_{i-1}$  (resp.  $TR'_{i-1}$ ) is connected to the input  $TR_{i-1}$  (resp.  $TR'_{i-1}$ ) of the cell on the left. The output  $TR_{i+1}$  (resp.  $TR'_{i+1}$ ) is connected to the input  $TR_{i+1}$  (resp.  $TR'_{i+1}$ ) of the cell on the right. The gadgets  $Right_{i-1}$  of the cell gadget  $Cell_{i-1}$  and  $Left_{i+1}$  of the cell gadget  $Cell_{i+1}$  are also depicted. Suppose that there is a robot  $r_i$  in the gadget  $Y_0^i$  and a robot  $h_1$  in  $TR_i$ . The two robots execute the reading operation by following red paths. In this example robots read the symbol  $Y_0$ . Now, suppose that there is a robot  $h_2$  in the gadget  $TR'_i$  and two robots  $r_i$  and  $h_1$  that come from the communication pipes  $CCP_{n-1}$  and  $CCP^R$ , respectively. The three robots execute the writing operation by following blue paths and the moving operation by following green and blue paths. In this example, robots write the symbol  $Y_{n-1}$  in the cell  $c_i$  and move the head to the right (i.e. to the cell  $c_{i+1}$ ). These paths are detailed in Lemma 1.

The role of this gadget is twofold:

- it changes the state of the machine, and
- it transmits to the current gadget cell the symbol to write and the direction in which the head has to move.

A robot  $s$  is used to encode the current state of  $M$ . That is,  $s$  is located in  $CR_i$  if and only if the current state of  $M$  is  $q_i$ . In that case, the controller gadget is said to be in the state  $q_i$ .

**Lemma 2.** *Consider a controller gadget produced by Construction 3. Suppose that  $s$  is in  $CR_i$ . Suppose that two robots  $r_j$  and  $h_1$  are coming from  $TCP_k$  and  $TCP^h$  respectively. Let  $(q_p, Y_\ell, D)$  (with  $D \in \{L, R\}$ ) be the values returned by  $\delta(q_i, Y_k)$ . Then,  $s$ ,  $r_j$  and  $h_1$  reach  $CR_p$ ,  $CCP_\ell$  and  $CCP^D$  respectively.*

*Proof.* By Property 1,  $s$  can intercept successively  $r_j$  and  $h_1$  in order to help them to reach  $\Delta_k^i$  and  $R'_{q_i}$ , respectively. By Construction 3, the robot  $r_j$  cannot reach another gadget than  $\Delta_k^i$ . Later, by Property 1,  $h_1$  and  $s$  can reach any  $\Delta_t^i$  gadget ( $\forall Y_t \in \Gamma$ ). If at least one of  $h_1$  and  $r_s$  reaches a gadget  $\Delta_t^i$  such that  $t \neq k$ , then by Property 2,  $s$ ,  $r_{c_j}$  and  $h_1$  are stuck in their gadget. Thus, suppose that both  $s$  and  $h_1$  enter in  $\Delta_k^i$ . By Property 2 and according to Construction 3,  $s$ ,  $r_j$  and  $h_1$  reach  $CR_p$ ,  $CCP_\ell$  and  $CCP^D$  respectively.

When the controller gadget receives the robots  $h_1$  and  $r_j$  from the tape gadget, it performs a *transiting operation* defined as below.

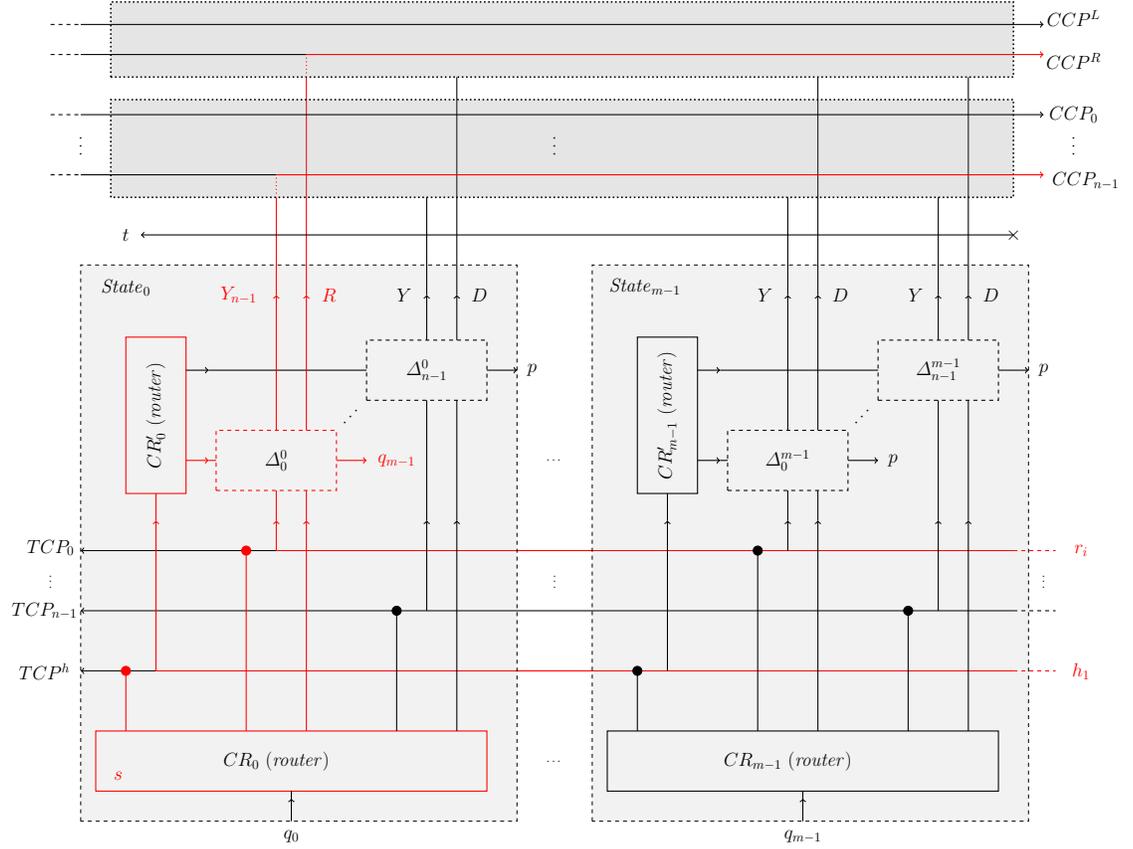
- **Transiting operation.** Suppose that there is a robot  $s$  in the router gadget  $CR_i$  and two robots  $r_j$  and  $h_1$  coming from  $TCP_\ell$  and  $TCP^h$  respectively. According to Lemma 2,  $r_j$  and  $h_1$  are sent to the tape gadget through  $CCP_\ell$  and  $CCP^D$  (with  $D \in \{L, R\}$ ) according to the transition function. Red paths in Figure 6 depict an example of transiting operation.

#### 4.4 Full construction

An instance of Ricochet Robots that simulates a Turing machine  $M$  is presented in this subsection.

**Construction 4** *Let  $M$  be a Turing machine with input  $\sigma$ . Let  $G$  be the board produced by Construction 1, Construction 2 and Construction 3. An instance of GENERALIZED REACHABILITY (GR)  $I = (B, R, T)$  is created as follows. Robots composing  $R$  have the following starting positions:*

- the robot  $s$  is placed in  $CR_0$  (in  $State_0$ , itself in the controller gadget),
- let cell  $c_i$  be the initial current cell, place two robots  $h_1$  and  $h_2$  in gadgets  $TR_i$  and  $TR'_i$  respectively (in the cell gadget  $Cell_i$ , itself in the tape gadget),
- for all cell  $c_i$  of the tape of  $M$ , let  $Y_j$  be the initial symbol of  $c_i$  in the input  $T$ , a robot  $r_i$  starts in the gadget  $Y_j^i$  (in  $Cell_i$ , itself in the tape gadget).



**Fig. 6.** The controller gadget that contains the unique target tile  $t$ . The connection on the dotted rectangles and the output  $p$  depend of the transition function  $\delta$ . For example, if  $\delta(q_0, Y_0) \rightarrow (q_{m-1}, Y_{n-1}, R)$  then the  $\Delta_0^0$  gadget is connected as follows: its output  $p$  is connected to the input  $q_{m-1}$  of the  $CR_{m-1}^{m-1}$  gadget, its output  $Y$  is connected to the communication pipe  $CCP_{n-1}$  and its output  $D$  to the pipe  $CCP^R$ . Suppose that there is a robot  $h_2$  in the gadget  $CR_0$  and two robots  $r_i$  and  $h_1$  coming from the communication pipes  $TCP_0$  and  $TCP^h$ , respectively. The three robots execute the transition function by following red paths (see Lemma 2 for more details).

The unique target tile  $t$  is located in the controller gadget (see Construction 3), hence  $T = t$ . A step in the instance produced by Construction 4 is composed of the following sequence of operations: (1) reading operation, (2) transitioning operation, (3) writing operation, and (4) moving operation. Lemma 1 and Lemma 2 ensure that the sequence order is respected and that no other operation is performed.

As said before, even if the instance produced by Construction 4 has an infinite horizontal dimension, it can be encoded in finite way as follows. For all  $\ell \in \Gamma$ ,  $C^\ell$  denotes a cell gadget  $Cell_i$  with a robot in  $TR_i^\ell$  and another one in  $Y_\ell^i$ . Thus, given an input  $\sigma \in \Sigma^* = \ell_1 \ell_2 \dots \ell_{|\sigma|}$  to  $M$ , the tape can be encoded

by the string  $((C^{Y_0})^* \circ C^{\ell_1} \circ \dots \circ C^{\ell_{|\sigma|}} \circ (C^{Y_0})^*)''$  (Recall that  $Y_0$  is the blank symbol). The controller gadget is then positioned above the first cell gadget that does not contain a blank character. Hence, the construction depicted here is a computational function taking a Turing machine as input and returning an instance of Ricochet Robots.

**Theorem 2.** *Considering an arbitrary Turing machine  $M$  and the corresponding instance  $I$  of GR obtained by Construction 4, for any  $k$ , at the  $k^{\text{th}}$  step in  $I$  and the  $k^{\text{th}}$  transition in  $M$ , the following conditions are respected:*

- (a) *the controller gadget is in the state  $q_i$  if and only if  $M$  is in state  $q_i$ ,*
- (b) *the current cell gadget is  $Cell_j$  if and only if the head of  $M$  is over the cell  $c_j$  of the tape, and*
- (c) *for all cell  $c_j$ , the cell gadgets  $Cell_j$  contains  $Y_\ell$  if and only if the cell  $c_j$  of the tape of  $M$  contains the symbol  $Y_\ell$ .*

*Proof.* Clearly (a), (b) and (c) are true if  $k = 0$ . Suppose that  $k > 0$  steps and transitions have been completed in  $I$  and  $M$  and that (a), (b) and (c) are verified. Let  $q_i$  be the current state,  $c_j$  be the current cell of the tape and  $Y_\ell$  be the symbol contained in  $c_j$ . Let  $(p, Y_t, D)$  (with  $D \in \{L, R\}$ ) be the values returned by  $\delta(q_i, Y_\ell)$ . The following shows that (a), (b) and (c) are still verified for the step  $k + 1$  in  $I$  and for the transition  $k + 1$  in  $M$ . First, by Lemma 1(1), a reading operation is performed in the current cell gadget,  $r_j$  and  $h_1$  are sent to the controller gadget through  $TCP_\ell$  and  $TCP^h$ , respectively. By Lemma 2, a transiting operation is then performed in the controller gadget. The robot  $s$  (that represents the current state) enters the state gadget  $State_p$  in the gadget  $CR_p$  while  $r_j$  and  $h_1$  are sent to the tape gadget through  $CCP_t$  and  $CCP^D$ , respectively. Further by Lemma 1, a writing operation and a moving operation are done in the tape gadget, that is,  $r_j$  reaches  $Y_t^j$  and  $h_1$  and  $h_2$  enter in gadgets  $TR_{j-1}$  and  $TR'_{j-1}$  (in the gadget  $Cell_{j-1}$ ) if  $D = L$ , or in  $TR_{j+1}$  and  $TR'_{j+1}$  (in  $Cell_{j+1}$ ) if  $D = R$ . Now,  $M$  is in state  $q_p$  and its head is over the cell  $c_{j-1}$  or  $c_{j+1}$  (depending on if  $D = L$  or  $D = R$ ). Hence, (a) and (b) are verified for  $k + 1$ . Moreover, the cell  $c_j$  contains  $Y_t$  in  $M$  and the robot is in the  $Y_t^j$  (in the gadget  $Cell_j$ ). Since only the cell  $c_j$  and the cell gadget  $Cell_j$  have been modified by the previous transition and step, (c) is verified for  $k + 1$ .

**Theorem 3.** *The instance  $I = (B, R, T)$  of GENERALIZED REACHABILITY (GR) problem obtained by Construction 4 admits a winning configuration if and only if  $M$  reaches a halting state.*

*Proof.* Suppose that  $I$  admits a winning configuration (*i.e.* a robot reaches the target tile). Thus, robots reach a gadget  $\Delta_j^i$  such that  $\delta(q_i, Y_j)$  is a halting case for  $M$ . Hence, according to Theorem 2, robots have executed a sequence of transitioning operations such that the corresponding sequence of transitions for  $M$  is a transition from the initial configuration to a halting case.

Suppose that  $M$  reaches a halting case  $\delta(q_i, Y_j)$ . Thus, there is a sequence of transitions from the initial configuration to a halting case in  $M$ . According to

Theorem 2, the corresponding sequence of transitioning operations allows robots of  $I$  to reach the gadget  $\Delta_j^i$ . Thus, one robot can reach the target tile  $t$ , then  $I$  admits a winning configuration.

By Theorem 3, the GENERALIZED REACHABILITY (GR) problem can simulate any Turing machine, then it can simulate a universal Turing machine, and then we prove the Theorem 1.

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