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NONREPETITIVE COLORINGS OF \mathbb{R}^d

(EXTENDED ABSTRACT)

Kathleen Barsse* Daniel Gonçalves† Matthieu Rosenfeld‡

Abstract

The results of Thue state that there exists an infinite sequence over 3 symbols without 2 identical adjacent blocks, which we call a 2-nonrepetitive sequence, and also that there exists an infinite sequence over 2 symbols without 3 identical adjacent blocks, which is a 3-nonrepetitive sequence. An r -repetition is defined as a sequence of symbols consisting of r identical adjacent blocks, and a sequence is said to be r -nonrepetitive if none of its subsequences are r -repetitions. Here, we study colorings of Euclidean spaces related to the work of Thue. A coloring of \mathbb{R}^d is said to be r -nonrepetitive if no sequence of colors derived from a set of collinear points at distance 1 is an r -repetition. In this case, the coloring is said to *avoid* r -repetitions. It was proved in [9] that there exists a coloring of the plane that avoids 2-repetitions using 18 colors, and conversely, it was proved in [3] that there exists a coloring of the plane that avoids 43-repetitions using only 2 colors. We specifically study r -nonrepetitive colorings for fixed number of colors : for a fixed number of colors k and dimension d , the aim is to determine the minimum multiplicity of repetition r such that there exists an r -nonrepetitive coloring of \mathbb{R}^d using k colors.

We prove that the plane, \mathbb{R}^2 , admits a 2- and a 3-coloring avoiding 33- and 18-repetitions, respectively.

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1 Introduction

The Hadwiger-Nelson problem asks for the minimum number of colors required to color the Euclidean plane such that any two points at distance 1 are colored differently. This is called the chromatic number of the plane, and is denoted as $\chi(\mathbb{R}^2)$. The answer to this problem is unknown, but it was proved that $5 \leq \chi(\mathbb{R}^2) \leq 7$ [1, 2, 7]. We study colorings of Euclidean spaces that are connected to the Hadwiger-Nelson problem and where the goal is to avoid specific patterns on straight lines.

An r -repetition is a finite sequence of symbols consisting of r identical blocks, where a *block* is a subsequence of consecutive terms. A sequence is r -nonrepetitive if none of its subsequences of consecutive terms are r -repetitions. For instance, the word *hotshots* is a 2-repetition and the word *minimize* is 2-nonrepetitive. The results of Thue state that there exists an infinite 2-nonrepetitive sequence over 3 symbols and an infinite 3-nonrepetitive sequence over 2 symbols. We study the Euclidean variant of Thue sequences introduced by Grytczuk *et al.* [5]. A *straight path* is defined as a sequence of collinear points of \mathbb{R}^d , where consecutive points are at distance 1. A coloring of \mathbb{R}^d is r -nonrepetitive if for each straight path in \mathbb{R}^d , the sequence of the colors of its points is r -nonrepetitive. For fixed integers d and r , the aim is to find the minimum number of colors for which there exists an r -nonrepetitive coloring of \mathbb{R}^d . Let $\pi_r(\mathbb{R}^d)$ denote that number.

One easily deduces from Thue's result that $\pi_2(\mathbb{R}) = 3$ and $\pi_3(\mathbb{R}) = 2$. The problem is more difficult for higher dimensions. Colorings of Euclidean spaces that avoid 2-repetitions are called *square-free* colorings. It was proven in [9] that there exists a square-free coloring of the plane that uses 18 colors, which means that $\pi_2(\mathbb{R}^2) \leq 18$. The problem of determining $\pi_2(\mathbb{R}^2)$ is connected to the Hadwiger-Nelson problem in the following way. If a coloring of the plane is 2-nonrepetitive, then 2 points at distance 1 must be colored differently, so at least $\chi(\mathbb{R}^2)$ colors are required. Therefore $5 \leq \chi(\mathbb{R}^2) \leq \pi_2(\mathbb{R}^2) \leq 18$.

Dębski *et al.* studied r -nonrepetitive colorings for larger values of r [3]. More specifically, they gave a proof that for any $d \in \mathbb{N}$, there exists $r = r(d)$ such that $\pi_r(\mathbb{R}^d) = 2$. In other words, for large enough values of r , the problem can be solved with the least possible number of colors. In particular, for $d = 2$, the minimum value of r for which $\pi_r(\mathbb{R}^2) = 2$ is unknown, but the paper provides a proof that $\pi_{43}(\mathbb{R}^2) = 2$ and $\pi_{24}(\mathbb{R}^2) \leq 3$. For smaller values of r , it is known that $\pi_6(\mathbb{R}^2) \leq 4$ and $\pi_3(\mathbb{R}^2) \leq 9$ [4, 9].

We prove that there exists a 33-nonrepetitive coloring of \mathbb{R}^2 with 2 colors, that is, $\pi_{33}(\mathbb{R}^2) = 2$. We also prove that $\pi_{18}(\mathbb{R}^2) \leq 3$. Our improvements rely on two main ingredients. First, we provide a better bound on the number of *pathable sequences of hypercubes*. This quantity already played a crucial role in the proof from [3]. Secondly, the proof from [3] uses the Lovász Local Lemma, which we replace with a counting method that yields slightly better bounds in this setting. This argument was first used for nonrepetitive colorings of graphs [6] and was later presented in the more general context of hypergraph coloring [8].

2 Pathable sequences

A standard technique in problems related to colorings of Euclidean spaces is to define a regular tiling of that space and assign the same color to all the points of each tile. The proof of the result from [3] uses a partition of \mathbb{R}^d into hypercubes of diameter 1. We will also use this partition. More precisely, each hypercube is a set of the form $\{(x_1, \dots, x_d) \in \mathbb{R}^d : \forall j \in \llbracket 1, d \rrbracket, i_j \leq x_j \sqrt{d} < i_j + 1\}$, with $(i_1, \dots, i_d) \in \mathbb{Z}^d$. This way, any two points at distance 1 are always in different hypercubes. Let \mathcal{H} denote the set of hypercubes from this partition.

We call a sequence $(\alpha_0, \dots, \alpha_{\ell-1})$ of hypercubes ℓ -pathable if there exists a straight path $(q_0, \dots, q_{\ell-1})$ in \mathbb{R}^d with $q_i \in \alpha_i$ for each i (See Figure 1). For a fixed cube H , $D_d(\ell)$ is defined as the number of ℓ -pathable sequences in \mathbb{R}^d containing H (each pair $(\alpha_0, \dots, \alpha_{\ell-1})$ and $(\alpha_{\ell-1}, \dots, \alpha_0)$ is counted as a single sequence).

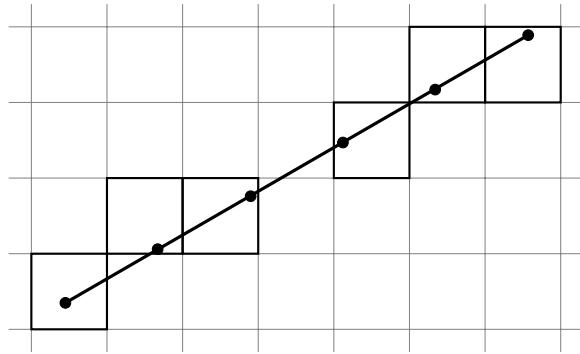


Figure 1: A 6-pathable sequence in \mathbb{R}^2 .

It is known that $D_d(\ell) = O(\ell^{3d})$ [3]. We improve this upper bound for $d = 2$.

Lemma 1. *The number of ℓ -pathable sequences in \mathbb{R}^2 is bounded as follows,*

$$D_2(\ell) \leq \frac{2\sqrt{2}}{3}\ell^5 + \left(2 - \frac{2\sqrt{2}}{3}\right)\ell^3 - 2\ell^2.$$

3 Calculations with the counting argument

In this section, we provide a condition similar to [3, Lemma 2.3]. It provides a condition on r and the number of colors k that ensures that there exists an r -nonrepetitive coloring of \mathbb{R}^d using k colors. However, the condition of Lemma 3, can be proven to be weaker, that is, whenever the condition of [3, Lemma 2.3] holds then our lemma automatically holds with $\beta = k2^{-1/(r-1)}$. In practice, this leads to a slightly better bound for our results.

In the proof of Lemma 2, we will consider an arbitrary subset S of \mathbb{R}^d consisting of finitely many hypercubes from the partition. This method directly shows that there exist exponentially many valid hypercube colorings, with respect to the number of hypercubes in S .

Lemma 2. *Let r, k and d be integers. For every set S of hypercubes, let $\mathcal{C}(S)$ be the set of r -nonrepetitive hypercube colorings of S with k colors.*

If there exists $\beta > 1$ such that

$$k \geq \beta + \sum_{s=1}^{\infty} D_d(rs) \times \beta^{1-(r-1)s},$$

then for every set S of n hypercubes of the partition of \mathbb{R}^d and for every hypercube $H \in S$,

$$|\mathcal{C}(S)| \geq \beta |\mathcal{C}(S - H)|.$$

Remark that $\beta > 1$ and that according to Corollary 2.5 from [3], $D_d(rs) = O((rs)^{3d})$, so the sum in this Lemma is always well-defined.

Proof. We proceed by induction on $n = |S|$. This is true for $n = 1$ because $S - H = \emptyset$. Fix $n \geq 2$ and assume that the result holds for every $i < n$. Let S be a set of n hypercubes and H a hypercube of S . Our induction hypothesis implies that for all $R \subseteq S - H$,

$$|\mathcal{C}(S - H - R)| \leq \frac{|\mathcal{C}(S - H)|}{\beta^{|R|}}. \tag{1}$$

Let F be the set of colorings of S that are r -nonrepetitive on $S - H$ but for which there is an r -repetition on S . Then

$$|\mathcal{C}(S)| = k |\mathcal{C}(S - H)| - |F|. \tag{2}$$

Let $s \in \mathbb{N}^*$ and $\alpha = (\alpha_1, \dots, \alpha_{rs})$ be a pathable sequence such that $H = \alpha_i$, for some $i \in \{1, \dots, rs\}$. We define F_α as the subset of F for which there is an r -repetition of length rs on that sequence. Without loss of generality, we assume that $i \geq s + 1$. We consider a coloring $\phi \in F_\alpha$. By definition of F , the sequence of colors on α is an r -repetition, and the restriction of ϕ to $S - (\alpha_{s+1}, \dots, \alpha_{rs})$ is r -nonrepetitive because $H \in \{\alpha_{s+1}, \dots, \alpha_{rs}\}$. Therefore, ϕ is uniquely determined by its restriction to $S - \{\alpha_{s+1}, \dots, \alpha_{rs}\}$ and $|F_\alpha| \leq |\mathcal{C}(S - \{\alpha_{s+1}, \dots, \alpha_{rs}\})|$. By equation (1), this implies,

$$|F_\alpha| \leq \frac{1}{\beta^{(r-1)s-1}} |\mathcal{C}(S - H)|.$$

Let F_{rs} be the subset of F for which there is an r -repetition of length rs . Recall that $D_d(rs)$ is the number of pathable sequences of length rs containing H . Then,

$$|F_{rs}| \leq D_d(rs) \frac{1}{\beta^{(r-1)s-1}} |\mathcal{C}(S - H)|.$$

Now, by summing over all s , and by using our main hypothesis

$$|F| = \left| \bigcup_{s=1}^{\infty} F_{rs} \right| \leq \sum_{s=1}^{\infty} |F_{rs}| \leq \sum_{s=1}^{\infty} D_d(rs) \frac{1}{\beta^{(r-1)s-1}} |\mathcal{C}(S - H)| \leq |\mathcal{C}(S - H)| (k - \beta).$$

Using this bound inside equation (2),

$$|\mathcal{C}(S)| = k|\mathcal{C}(S - H)| - |F| \geq \beta|\mathcal{C}(S - H)|$$

which concludes our induction. \square

For each subset S of \mathbb{R}^d consisting of n hypercubes, $|\mathcal{C}(S)| \geq \beta^{n-1}k$. This means that any finite arbitrary subset of hypercubes of the partition of \mathbb{R}^d can be r -nonrepetitively colored. By compactness (e.g., see the proof of Lemma 2.3 from [3]) there exists an r -nonrepetitive coloring of \mathbb{R}^d .

Lemma 3. *For every integers r , k and d , if there exists $\beta > 1$ such that*

$$k \geq \beta + \sum_{s=1}^{\infty} D_d(rs) \times \beta^{1-(r-1)s}$$

then $\pi_r(\mathbb{R}^d) \leq k$.

4 Proof of the main results and conclusion

We can now use the bound from Lemma 1 to verify the conditions of Lemma 3 for well-chosen values of r , β and k . In particular, one can verify that the condition of Lemma 3 holds for $r = 33$, $\beta = 19/10$ and $k = 2$ which implies the following result.

Theorem 4. *There exists a 2-coloring of the plane avoiding 33-repetitions.*

Let $r(d)$ denote the least positive integer such that $\pi_{r(d)}(\mathbb{R}^d) = 2$. We proved that $r(2) \leq 33$, which improves the bound $r(2) \leq 43$ proved in [3]. However, this result probably isn't optimal, since the best known lower bound is $r(2) \geq 3$ which is a consequence of the results of Thue. This means that $r(2)$ lies between 3 and 33. In fact, it is conjectured in [3] that $r(2) = 4$.

Similarly, one can verify that the condition of Lemma 3 holds for $r = 18$, $\beta = 8/3$ and $k = 3$ which implies the following result.

Theorem 5. *There exists a 3-coloring of the plane avoiding 18-repetitions.*

Again the value 18 is an improvement from 24 but is probably still not optimal.

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