Critical Exponent of Binary Words with Few Distinct Palindromes

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Submitted: Nov 21, 2023; Accepted: Apr 30, 2024; Published: May 17, 2024

Abstract

We study infinite binary words that contain few distinct palindromes. In particular, we classify such words according to their critical exponents. This extends results by Fici and Zamboni [TCS 2013]. Interestingly, the words with 18 and 20 palindromes happen to be morphic images of the fixed point of the morphism $\mathtt{0} \mapsto \mathtt{01}$, $\mathtt{1} \mapsto \mathtt{21}$, $\mathtt{2} \mapsto \mathtt{0}$.

Mathematics Subject Classifications: 68R15

1 Introduction

We consider the trade-off between the number of distinct palindromes and the critical exponent in infinite binary words. For brevity, every mention of a number of palindromes will refer to a number of distinct palindromes, including the empty word. Fici and Zamboni [6] show that the least number of palindromes in an infinite binary word is 9 and this bound is reached by the word $(\mathtt{001011})^\omega$. At the other end of the spectrum, the famous Thue-Morse word TM, fixed point of the morphism $\mathtt{0} \to \mathtt{01}$, $\mathtt{1} \to \mathtt{10}$, has the least critical exponent and infinitely many palindromes.

Our results completely answer questions of this form: do infinite β^+-free binary words with at most p palindromes exist? In each case, we also determine whether there are exponentially or polynomially many such words. The results are summarized in Table 1. A green (resp. red) cell means that there are exponentially (resp. polynomially) many words. We have labelled the cells that correspond to an item of Theorem 3 or 7.

Fici and Zamboni [6] also show that an aperiodic binary word contains at least 11 palindromes and this bound is reached by the morphic image of the Fibonacci word by $\mathtt{0} \to \mathtt{0}$, $\mathtt{1} \to \mathtt{01101}$. This word contains in particular the factor

$$(\mathtt{0011010001101001101000110100011010011010001101001101000110100011010011010001101000110 1})^{\frac{7}{2}}.$$

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Theorem 1.(a) improves this exponent to $\frac{10}{3}^+$. Fleischer and Shallit [7] have considered the number of binary words of length n with at most 11 palindromes (sequence A330127 in the OEIS) and proved that it is $\Theta(\kappa^n)$, where $\kappa = 1.1127756842787\ldots$ is the root of $X^7 = X + 1$.

p \ β^+	2^+	$\frac{7}{3}^+$	$\frac{5}{2}^+$	$\frac{28}{11}^+$	$\frac{13}{5}^+$	$\frac{8}{3}^+$	3^+	$\frac{23}{7}^+$	$\frac{10}{3}^+$	∞
∞	TM									
25		3.h								
24										
23										
22										
21			3.g							
20			7.b							
19				3.f						
18				7.a	3.e					
17										
16										
15						3.d				
14										
13							3.c			
12								3.b		
11									3.a	
10										
9										$(001011)^\omega$

Table 1: Infinite β^+-free binary words with at most p palindromes.

2 Preliminaries

An *alphabet* $\mathcal{A}$ is a finite set and its elements are called *letters*. A *word* u over $\mathcal{A}$ of *length* n is a finite string $u = u_0u_1 \cdots u_{n-1}$, where $u_j \in \mathcal{A}$ for all $j \in \{0, 1, \ldots, n-1\}$. If $\mathcal{A} = \{\mathtt{0}, \mathtt{1}, \ldots, \mathtt{d}-\mathtt{1}\}$, the length of u is denoted $|u|$ and $|u|_\mathtt{i}$ denotes the number of occurrences of the letter $\mathtt{i} \in \mathcal{A}$ in the word u. The *Parikh vector* $\vec{u} \in \mathbb{N}^d$ is the vector defined as $\vec{u} = (|u|_\mathtt{0}, |u|_\mathtt{1}, \ldots, |u|_{\mathtt{d-1}})^T$. The set of all finite words over $\mathcal{A}$ is denoted $\mathcal{A}^*$. The set $\mathcal{A}^*$ equipped with concatenation as the operation forms a monoid with the *empty word* ε as the neutral element. We will also consider the set $\mathcal{A}^\omega$ of infinite words (that is, right-infinite words) and the set ${}^\omega\mathcal{A}^\omega$ of bi-infinite words. A word v is an *e-power* of a word u if v is a prefix of the infinite periodic word $uuu\cdots = u^\omega$ and $e = |v|/|u|$. We write $v = u^e$. We also call u^e a repetition with period u and exponent e. For instance, the Czech word *kapka* (drop) can be written in this formalism as $(kap)^{5/3}$. A word is *α^+-free* (resp. *α-free*) if it contains no repetition with exponent β such that $\beta > \alpha$ (resp. $\beta \geqslant \alpha$).

The *critical exponent* $E(\mathbf{u})$ of an infinite word $\mathbf{u}$ is defined as

$$E(\mathbf{u}) = \sup\{e \in \mathbb{Q} : \ u^e \text{ is a factor of } \mathbf{u} \text{ for a non-empty word } u\}\,.$$

The *asymptotic critical exponent* $E^*(\mathbf{u})$ of an infinite word $\mathbf{u}$ is defined as $+\infty$ if $E(\mathbf{u}) = +\infty$, and

$$E^*(\mathbf{u}) = \limsup_{n\to\infty}\{e \in \mathbb{Q} : \ u^e \text{ is a factor of } \mathbf{u} \text{ for some } u \text{ of length } n\}\,,$$

otherwise. If each factor of $\mathbf{u}$ has infinitely many occurrences in $\mathbf{u}$, then $\mathbf{u}$ is *recurrent*. Moreover, if for each factor the distances between its consecutive occurrences are bounded, then $\mathbf{u}$ is *uniformly recurrent*. The *language* $\mathcal{L}(\mathbf{u})$ is the set of factors occurring in $\mathbf{u}$. The language $\mathcal{L}(\mathbf{u})$ is *closed under reversal* if for each factor $w = w_0 w_1 \cdots w_{n-1}$, its *reverse* $w^R = w_{n-1} \cdots w_1 w_0$ is also a factor of $\mathbf{u}$. A word w is a *palindrome* if $w = w^R$. Let us denote $\overline{\mathtt{0}} = \mathtt{1}$ and $\overline{\mathtt{1}} = \mathtt{0}$, then for any binary word w its *bit complement* is $\overline{w} = \overline{w_0}\ \overline{w_1} \cdots \overline{w_{n-1}}$.

Consider a factor w of a recurrent infinite word $\mathbf{u} = u_0 u_1 u_2 \cdots$. Let $j < \ell$ be two consecutive occurrences of w in $\mathbf{u}$. Then the word $u_j u_{j+1} \cdots u_{\ell-1}$ is a *return word* to w in $\mathbf{u}$.

The *(factor) complexity* of an infinite word $\mathbf{u}$ is the mapping $\mathcal{C}_\mathbf{u} : \mathbb{N} \to \mathbb{N}$ defined by $\mathcal{C}_\mathbf{u}(n) = \#\{w \in \mathcal{L}(\mathbf{u}) : |w| = n\}$.

Given a word $w \in \mathcal{L}(\mathbf{u})$, we define the sets of left extensions, right extensions and bi-extensions of w in $\mathbf{u}$ over an alphabet $\mathcal{A}$ respectively as

$$L_\mathbf{u}(w) = \{\mathtt{i} \in \mathcal{A} : \mathtt{i}w \in \mathcal{L}(\mathbf{u})\}, \qquad R_\mathbf{u}(w) = \{\mathtt{j} \in \mathcal{A} : w\mathtt{j} \in \mathcal{L}(\mathbf{u})\}$$

and

$$B_\mathbf{u}(w) = \{(\mathtt{i},\mathtt{j}) \in \mathcal{A} \times \mathcal{A} : \mathtt{i}w\mathtt{j} \in \mathcal{L}(\mathbf{u})\}.$$

If $\#L_\mathbf{u}(w) > 1$, then w is called *left special (LS)*. If $\#R_\mathbf{u}(w) > 1$, then w is called *right special (RS)*. If w is both LS and RS, then it is called *bispecial (BS)*. We define $b(w) = \#B_\mathbf{u}(w) - \#L_\mathbf{u}(w) - \#R_\mathbf{u}(w) + 1$ and we distinguish *ordinary BS factors* with $b(w) = 0$, *weak BS factors* with $b(w) < 0$ and *strong BS factors* with $b(w) > 0$.

A *morphism* is a map $\psi : \mathcal{A}^* \to \mathcal{B}^*$ such that $\psi(uv) = \psi(u)\psi(v)$ for all words $u, v \in \mathcal{A}^*$. The morphism ψ is *non-erasing* if $\psi(\mathtt{i}) \neq \varepsilon$ for each $\mathtt{i} \in \mathcal{A}$. Morphisms can be naturally extended to infinite words by setting $\psi(u_0 u_1 u_2 \cdots) = \psi(u_0)\psi(u_1)\psi(u_2)\cdots$. A *fixed point* of a morphism $\psi : \mathcal{A}^* \to \mathcal{A}^*$ is an infinite word $\mathbf{u}$ such that $\psi(\mathbf{u}) = \mathbf{u}$. We associate to a morphism $\psi : \mathcal{A}^* \to \mathcal{A}^*$ the *(incidence) matrix* M_ψ defined for each $k, j \in \{0, 1, \ldots, d-1\}$ as $[M_\psi]_{kj} = |\psi(\mathtt{j})|_\mathtt{k}$.

If there exists $N \in \mathbb{N}$ such that M_ψ^N has positive entries, then ψ is a *primitive* morphism. By definition, we have for each $u \in \mathcal{A}^*$ the following relation for the Parikh vectors $\vec{\psi(u)} = M_\psi \vec{u}$.

Let $\mathbf{u}$ be an infinite word over an alphabet $\mathcal{A}$. Then the *uniform frequency* $f_\mathtt{i}$ of the letter $\mathtt{i} \in \mathcal{A}$ is equal to α if for any sequence (w_n) of factors of $\mathbf{u}$ with increasing lengths

$$\alpha = \lim_{n\to\infty} \frac{|w_n|_\mathtt{i}}{|w_n|}\,.$$

It is known that fixed points of primitive morphisms have uniform letter frequencies [9].

Let $\mathbf{u}$ be an infinite word over an alphabet $\mathcal{A}$ and let $\psi : \mathcal{A}^* \to \mathcal{B}^*$ be a morphism. Consider a factor w of $\psi(\mathbf{u})$. We say that (w_1, w_2) is a *synchronization point* of w if $w = w_1w_2$ and for all $p, s \in \mathcal{L}(\psi(\mathbf{u}))$ and $v \in \mathcal{L}(\mathbf{u})$ such that $\psi(v) = pws$ there exists a factorization $v = v_1v_2$ of v with $\psi(v_1) = pw_1$ and $\psi(v_2) = w_2s$. We denote the synchronization point by $w_1 \bullet w_2$.

Given a factorial language L and an integer ℓ, let L^ℓ denote the words of length ℓ in L. The *Rauzy graph* of L of order ℓ is the directed graph whose vertices are the words of $L^{\ell-1}$, the arcs are the words of L^ℓ, and the arc corresponding to the word w goes from the vertex corresponding to the prefix of w of length $\ell - 1$ to the vertex corresponding to the suffix of w of length $\ell - 1$.

Finally, this paper mainly studies properties of the words $\mu(\mathbf{p})$ and $\nu(\mathbf{p})$ that are morphic images of the word $\mathbf{p} = \varphi^\omega(0)$ studied in [2], where

$$\begin{array}{lll} \varphi(0) = 01 & \mu(0) = 011001 & \nu(0) = 011 \\ \varphi(1) = 21 & \mu(1) = 1001 & \nu(1) = 0 \\ \varphi(2) = 0 & \mu(2) = 0 & \nu(2) = 01 \end{array}$$

3 Fewest palindromes, least critical exponent, and factor complexity

3.1 General result

Theorem 1. *There exists an infinite binary β^+-free word containing only p palindromes for the following pairs (p, β). Moreover, this list of pairs is optimal.*

(a) $(11, \frac{10}{3})$

(b) $(12, \frac{23}{7})$

(c) $(13, 3)$

(d) $(15, \frac{8}{3})$

(e) $(18, \frac{28}{11})$

(f) $(20, \frac{5}{2})$

(g) $(25, \frac{7}{3})$

Proof. The optimality is obtained by backtracking. For example, to obtain the step between items (c) and (d), we show that there exists no infinite cubefree word containing at most 14 palindromes. The proof of the positive results is split in two cases, depending on the factor complexity of the considered words, see Theorems 3 and 7. Let us already remark that, in any case, it is easy to check that the proposed word does not contain more than the claimed number of palindromes since it only requires to check the factors up to some finite length. ☐

3.2 Exponential cases

We need some terminology and a lemma from [11]. A morphism $f : \Sigma^* \to \Delta^*$ is *q-uniform* if $|f(a)| = q$ for every $a \in \Sigma$, and is called *synchronizing* if for all $a, b, c \in \Sigma$ and $u, v \in \Delta^*$, if $f(ab) = uf(c)v$, then either $u = \varepsilon$ and $a = c$, or $v = \varepsilon$ and $b = c$.

Lemma 2. *[11, Lemma 23] Let $a, b \in \mathbb{R}$ satisfy $1 < a < b$. Let $\alpha \in \{a, a^+\}$ and $\beta \in \{b, b^+\}$. Let $h\colon \Sigma^* \to \Delta^*$ be a synchronizing q-uniform morphism. Set*

$$t = \max\left(\frac{2b}{b-a}, \frac{2(q-1)(2b-1)}{q(b-1)}\right).$$

If $h(w)$ is β-free for every α-free word w with $|w| \leqslant t$, then $h(z)$ is β-free for every α-free word $z \in \Sigma^*$.

The results in this subsection use the following steps. We find an appropriate uniform synchronizing morphism h by exhaustive search. We use Theorem 2 to show that h maps every binary $\frac{7}{3}^+$-free word (resp. ternary squarefree word) to a suitable binary β^+-free word. Since there are exponentially many binary $\frac{7}{3}^+$-free words [10] (resp. ternary squarefree words [12]), there are also exponentially many binary β^+-free words.

Theorem 3. *There exist exponentially many infinite binary β^+-free words containing at most p palindromes for the following pairs (p, β).*

(a) $(11, \frac{10}{3})$

(b) $(12, \frac{23}{7})$

(c) $(13, 3)$

(d) $(15, \frac{8}{3})$

(e) $(18, \frac{13}{5})$

(f) $(19, \frac{28}{11})$

(g) $(21, \frac{5}{2})$

(h) $(25, \frac{7}{3})$

Proof.

(a) $(11, \frac{10}{3})$: Applying the 39-uniform morphism

$$0 \to 001011001011100101110010110010111001011$$
$$1 \to 100101100101100101110010110010111001011$$

to any binary $\frac{7}{3}^+$-free word gives a $\frac{10}{3}^+$-free binary word containing at most 11 palindromes.

(b) $(12, \frac{23}{7})$: Applying the 45-uniform morphism

$$0 \to \mathtt{000101100010111000101110001011000101110001011}$$
$$1 \to \mathtt{100010110001011000101110001011000101110001011}$$

to any binary $\frac{7}{3}^+$-free word gives a $\frac{23}{7}^+$-free binary word containing at most 12 palindromes.

(c) $(13, 3)$: Applying the 7-uniform morphism

$$0 \to \mathtt{0001011}$$
$$1 \to \mathtt{1001011}$$

to any binary $\frac{7}{3}^+$-free word gives a cubefree binary word containing at most 13 palindromes.

(d) $(15, \frac{8}{3})$: Applying the 3-uniform morphism

$$0 \to \mathtt{001}$$
$$1 \to \mathtt{101}$$

to any binary $\frac{7}{3}^+$-free word gives a $\frac{8}{3}^+$-free binary word containing at most 15 palindromes.

(e) $(18, \frac{13}{5})$: Applying the 72-uniform morphism

$$0 \to \mathtt{001011001100101100101001011001100101100110010100101100101001011001100101}$$
$$1 \to \mathtt{100110010100101100110010110010100101100110010100101100101001011001100101}$$

to any binary $\frac{7}{3}^+$-free word gives a $\frac{13}{5}^+$-free binary word containing at most 18 palindromes.

(f) $(19, \frac{28}{11})$: Applying the 49-uniform morphism

$$0 \to \mathtt{0010110010110101100101001011001010010110101100101}$$
$$1 \to \mathtt{1010110010100101100101101011001010010110101100101}$$

to any binary $\frac{7}{3}^+$-free word gives a $\frac{28}{11}^+$-free binary word containing at most 19 palindromes.

(g) $(21, \frac{5}{2})$: Applying the 10-uniform morphism

$$0 \to \mathtt{0011001101}$$
$$1 \to \mathtt{1001011001}$$

to any binary $\frac{7}{3}^+$-free word gives a $\frac{5}{2}^+$-free binary word containing at most 21 palindromes.

(h) $(25, \frac{7}{3})$: Applying the 36-uniform morphism

$$0 \to \texttt{001101100101100110110010011001011001}$$
$$1 \to \texttt{101100100110100110110010011001011001}$$
$$2 \to \texttt{001101100110100110110010011001011001}$$

to any ternary squarefree word gives a $\frac{7}{3}^+$-free binary word containing at most 25 palindromes. $\square$

3.3 Polynomial cases

Theorem 4. *[2] Every bi-infinite ternary cubefree word avoiding*

$$F = \{\texttt{00,11,22,20,212,0101,02102,121012,01021010,21021012102}\}$$

has the same set of factors as $\mathbf{p}$.

Lemma 5. *Every bi-infinite cubefree binary word avoiding*

$$F_{18} = \{\texttt{1101,00100,10101,010011,1011001011,110010110011,1011001010010110010}\}$$

has the same set of factors as $\mu(\mathbf{p})$.

Proof. Consider a bi-infinite binary cubefree word $\mathbf{w}$ avoiding F_{18}. The factors of $\mathbf{w}$ of length at least 5 that contain 0101 only as a prefix and a suffix are 01011001100101, 010100101, and 0101100101. Thus, $\mathbf{w}$ is in $\{\texttt{0110011001}, \texttt{01001}, \texttt{011001}\}^{\omega}$. So, $\mathbf{w}$ is in $\{\texttt{011001}, \texttt{1001}, \texttt{0}\}^{\omega}$. That is, $\mathbf{w} = \mu(\mathbf{v})$ for some bi-infinite ternary word $\mathbf{v}$. Since $\mathbf{w}$ is cubefree, its pre-image $\mathbf{v}$ is also cubefree.

To show that $\mathbf{v}$ avoids F, we consider every $f \in F$ and we show by contradiction that f is not a factor of $\mathbf{v}$.

(a) If $\mathbf{v}$ contains 22, then $\mu(\texttt{220}) = \texttt{00011001}$ and $\mu(\texttt{222}) = \texttt{000}$ contain $\texttt{0}^3$ and $\mu(\texttt{221}) = \texttt{001001}$ contains $\texttt{00100} \in F_{18}$.

(b) If $\mathbf{v}$ contains 20, then $\mathbf{v}$ contains $x\texttt{20}$ for $x \in \{\texttt{0}, \texttt{1}\}$ by (a).
$\mu(x\texttt{20})$ contains 10010011001 as a suffix, which contains $\texttt{00100} \in F_{18}$.

(c) If $\mathbf{v}$ contains 00, then $\mathbf{v}$ contains 100 to avoid the cube 000 and by (b).
$\mu(\texttt{100}) = \texttt{1001011001011001}$ contains $\texttt{1011001011} \in F_{18}$.

(d) If $\mathbf{v}$ contains 11, then $\mu(\texttt{011})$ and $\mu(\texttt{111})$ contain $(\texttt{1001})^3$ and $\mu(\texttt{211}) = \texttt{010011001}$ contains $\texttt{010011} \in F_{18}$.

(e) If $\mathbf{v}$ contains 212, then $\mathbf{v}$ contains 2121 by (a) and (b).
$\mathbf{v}$ contains $x\texttt{2121}y$ with $x \in \{\texttt{0}, \texttt{1}\}$ by (a) and $y \in \{\texttt{0}, \texttt{2}\}$ by (d).
Since $\mu(\texttt{1})$ is a suffix of $\mu(\texttt{0})$ and $\mu(\texttt{2})$ is a prefix of $\mu(\texttt{0})$, then $\mu(x\texttt{2121}y)$ contains the factor $\mu(\texttt{121212}) = \mu\left((\texttt{12})^3\right)$.

(f) If $\mathbf{v}$ contains 0101, then $\mu(0101) = 01100110010110011001$ contains $110010110011 \in F_{18}$.

(g) If $\mathbf{v}$ contains 02102, then $\mathbf{v}$ contains 102102 by (b) and (c).
$\mu(102102) = 10010110010100101100 10$ contains $1011001010010110010 \in F_{18}$.

(h) If $\mathbf{v}$ contains 121012, then $\mathbf{v}$ contains 0121012 by (d) and (e).
$\mathbf{v}$ contains 10121012 by (b) and (c).
$\mathbf{v}$ contains 210121012 by (d) and (f).
$\mathbf{v}$ contains 2101210121 by (a) and (b).
$\mathbf{v}$ contains 21012101210 by (d) and (e).
$\mathbf{v}$ contains $x21012101210$ with $x \in \{0, 1\}$ by (a).
Since $\mu(1)$ is a suffix of $\mu(0)$, then $\mu(x21012101210)$ contains
$\mu(121012101210) = \mu\left((1210)^3\right)$.

(i) If $\mathbf{v}$ contains 01021010, then $\mathbf{v}$ contains 010210102 by (c) and (f).
$\mathbf{v}$ contains 0102101021 by (a) and (b).
$\mathbf{v}$ contains 01021010210 by (d) and (e).
$\mathbf{v}$ contains 010210102101 by (c) and (g).
$\mathbf{v}$ contains 1010210102101 by (b) and (c).
$\mathbf{v}$ contains $21010210102101 = (21010)^2 2101$ by (d) and (f).
$\mathbf{v}$ contains $(21010)^2 21012$ by (d) and to avoid $(21010)^3$.
$\mathbf{v}$ contains $1(21010)^2 21012$ by (a) and to avoid $(02101)^3$.
$\mathbf{w}$ contains $\mu(1(21010)^2 21012) = 1001(\mu(210)1001011001)^2 \mu(210)10010$.
To avoid $00100 \in F_{18}$, $\mathbf{w}$ contains
$1001(\mu(210)1001011001)^2 \mu(210)100101 = (1001\mu(210)100101)^3$.

(j) If $\mathbf{v}$ contains 21021012102, then $\mathbf{v}$ contains 121021012102 by (a) and (g).
$\mathbf{v}$ contains 0121021012102 by (d) and (e).
$\mathbf{v}$ contains 10121021012102 by (b) and (c).
$\mathbf{v}$ contains 210121021012102 by (d) and (f).
$\mathbf{v}$ contains 0210121021012102 by (a) and (h).
$\mathbf{v}$ contains 10210121021012102 by (b) and (c).
$\mathbf{v}$ contains 102101210210121021 by (a) and (b).
$\mathbf{v}$ contains 1021012102101210210 by (d) and (e).
$\mathbf{v}$ contains 10210121021012102101 by (c) and (g).
$\mathbf{v}$ contains 102101210210121021010 by (d) and to avoid $(1021012)^3$.
$\mu(102101210210121021010) = (\mu(102101)0)^3 11001$. □

Lemma 6. *Every bi-infinite cubefree binary word avoiding*

$$F_{20} = \{0101, 1011, 010010, 1100110100110011\}$$

has the same set of factors as $\nu(\mathbf{p})$.

Proof. Consider a bi-infinite binary cubefree word $\mathbf{w}$ avoiding F_{20}. Since $\mathbf{w}$ is cubefree, $\mathbf{w}$ is in $\{\mathtt{011}, \mathtt{0}, \mathtt{01}\}^\omega$. So $\mathbf{w} = \nu(\mathbf{v})$ for some bi-infinite ternary word $\mathbf{v}$. Since $\mathbf{w}$ is cubefree, its pre-image $\mathbf{v}$ is also cubefree.

To show that $\mathbf{v}$ avoids F, we consider every $f \in F$ and we show by contradiction that f is not a factor of $\mathbf{v}$.

(a) If $\mathbf{v}$ contains 00, then $\nu(\mathtt{00}) = \mathtt{011011}$ contains $\mathtt{1011} \in F_{20}$.

(b) If $\mathbf{v}$ contains 11, then $\mathbf{v}$ contains $\mathtt{11}y$ for some letter y.
$\nu(\mathtt{11}y)$ contains the cube 000 as a prefix.

(c) If $\mathbf{v}$ contains 22, then $\nu(\mathtt{22}) = \mathtt{0101} \in F_{20}$.

(d) If $\mathbf{v}$ contains 20, then $\nu(\mathtt{20}) = \mathtt{01011}$ contains $\mathtt{0101} \in F_{20}$.

(e) If $\mathbf{v}$ contains 212, then $\mathbf{v}$ contains 2121 by (c) and (d).
$\nu(\mathtt{2121}) = \mathtt{010010} \in F_{20}$.

(f) If $\mathbf{v}$ contains 0101, then $\mathbf{v}$ contains 10101 by (a) and (d).
$\mathbf{v}$ contains 210101 by (b) and to avoid $(\mathtt{01})^3$.
$\mathbf{v}$ contains 2101012 by (b) and to avoid $(\mathtt{10})^3$.
$\nu(\mathtt{2101012}) = \mathtt{0100110011001} = \mathtt{0}(\mathtt{1001})^3$.

(g) If $\mathbf{v}$ contains 02102, then $\mathbf{v}$ contains 102102 by (a) and (d).
$\mathbf{v}$ contains 1021021 by (c) and (d).
$\mathbf{v}$ contains 10210210 by (b) and (e).
$\mathbf{w}$ contains $\nu(\mathtt{10210210}) = \mathtt{0011010011010011}$.
To avoid $\mathtt{0}^3$ and $\mathtt{1}^3$, $\mathbf{w}$ contains $\mathtt{100110100110100110} = (\mathtt{100110})^3$.

(h) If $\mathbf{v}$ contains 121012, then $\mathbf{v}$ contains 0121012 by (b) and (e).
$\mathbf{v}$ contains 10121012 by (a) and (d).
$\mathbf{v}$ contains 210121012 by (b) and (f).
$\mathbf{v}$ contains 2101210121 by (c) and (d).
$\mathbf{v}$ contains 21012101210 by (b) and (e).
$\mathbf{w}$ contains $\nu(\mathtt{21012101210}) = \mathtt{0100110010011001001 1}$.
To avoid $\mathtt{1}^3$, $\mathbf{w}$ contains $\mathtt{010011001001100100110} = (\mathtt{0100110})^3$.

(i) If $\mathbf{v}$ contains 01021010, then
$\nu(\mathtt{01021010}) = \mathtt{01100110100110011}$ contains $\mathtt{1100110100110011} \in F_{20}$.

(j) If $\mathbf{v}$ contains 21021012102, then $\mathbf{v}$ contains 121021012102 by (c) and (g).
$\mathbf{v}$ contains 0121021012102 by (b) and (e).
$\mathbf{v}$ contains 10121021012102 by (a) and (d).
$\mathbf{v}$ contains 210121021012102 by (b) and (f).
$\mathbf{v}$ contains 0210121021012102 by (c) and (h).
$\mathbf{v}$ contains 10210121021012102 by (a) and (d).
$\mathbf{v}$ contains 102101210210121021 by (c) and (d).

$\mathbf{v}$ contains 1021012102101210210 by (b) and (e).
$\mathbf{v}$ contains 10210121021012102101 by (a) and (g).
$\mathbf{v}$ contains 102101210210121021010 by (b) and to avoid $(1021012)^3$.
$\nu(102101210210121021010) = (0011010011001)^3 1$. □

Theorem 7.

(a) *The word $\mu(\mathbf{p})$ is $\frac{28}{11}^+$-free and contains 18 palindromes. Every bi-infinite $\frac{13}{5}$-free binary word containing at most 18 palindromes has the same set of factors as either $\mu(\mathbf{p})$, $\overline{\mu(\mathbf{p})}$, $\mu(\mathbf{p})^R$, or $\overline{\mu(\mathbf{p})}^R$.*

(b) *The word $\nu(\mathbf{p})$ is $\frac{5}{2}^+$-free and contains 20 palindromes. Every recurrent $\frac{28}{11}$-free binary word containing at most 20 palindromes has the same set of factors as either $\nu(\mathbf{p})$, $\overline{\nu(\mathbf{p})}$, $\nu(\mathbf{p})^R$, or $\overline{\nu(\mathbf{p})}^R$.*

Proof.

(a) We prove in Section 4.3 that $\mu(\mathbf{p})$ is $\frac{28}{11}^+$-free.

We construct the set S^{20}_{18} defined as follows: a word v is in S^{20}_{18} if and only if there exists a $\frac{13}{5}$-free binary word pvs containing at most 18 palindromes and such that $|p| = |v| = |s| = 20$. From S^{20}_{18}, we construct the Rauzy graph R^{20}_{18} such that the vertices are the factors of length 19 and the arcs are the factors of length 20. We notice that R^{20}_{18} is disconnected. It contains four connected components that are symmetric with respect to reversal and bit complement. Let C^{20}_{18} be the connected component which avoids the factor 1101. We check that C^{20}_{18} is identical to the Rauzy graph of the factors of length 19 and 20 of $\mu(\mathbf{p})$.

Now we consider a bi-infinite $\frac{13}{5}$-free binary word $\mathbf{w}$ with 18 palindromes. So $\mathbf{w}$ corresponds to a walk in one of the connected components of R^{20}_{18}, say C^{20}_{18} without loss of generality. By the previous remark, $\mathbf{w}$ has the same set of factors of length 20 as $\mu(\mathbf{p})$. Since $\max\{|f|, f \in F_{18}\} = 19 \leqslant 20$, $\mathbf{w}$ avoids every factor in F_{18}. Moreover, $\mathbf{w}$ is cubefree since it is $\frac{13}{5}$-free. By Theorem 5, $\mathbf{w}$ has the same factor set as $\mu(\mathbf{p})$.

Then the proof is complete by symmetry by reversal and bit complement.

(b) We prove in Section 4.2 that $\nu(\mathbf{p})$ is $\frac{5}{2}^+$-free.

We construct the set S^{78}_{20} defined as follows: a word v is in S^{78}_{20} if and only if there exists a $\frac{28}{11}$-free binary word pvs containing at most 20 palindromes and such that $|p| = |v| = |s| = 78$. From S^{78}_{20}, we construct the Rauzy graph R^{78}_{20} such that the vertices are the factors of length 77 and the arcs are the factors of length 78. We notice that R^{78}_{20} is not strongly connected. It contains four strongly connected components that are symmetric with respect to reversal and bit complement. Let C^{78}_{20} be the strongly connected component which avoids the factor 1011. We check that C^{78}_{20} is identical to the Rauzy graph of the factors of length 77 and 78 of $\nu(\mathbf{p})$.

Now we consider a recurrent $\frac{28}{11}$-free binary word $\mathbf{w}$ with 20 palindromes. Since $\mathbf{w}$ is recurrent, $\mathbf{w}$ corresponds to a walk in one of the strongly connected components of R_{20}^{78}, say C_{20}^{78} without loss of generality. By the previous remark, $\mathbf{w}$ has the same set of factors of length 78 as $\nu(\mathbf{p})$. Since $\max\{|f|, f \in F_{20}\} = 16 \leqslant 78$, $\mathbf{w}$ avoids every factor in F_{20}. Moreover, $\mathbf{w}$ is cubefree since it is $\frac{28}{11}$-free. By Theorem 6, $\mathbf{w}$ has the same factor set as $\nu(\mathbf{p})$.

Then the proof is complete by symmetry by reversal and bit complement. □

Notice that item (b) requires recurrent words rather than bi-infinite words. That is because of, e.g., the bi-infinite word $\mathbf{x} = \nu(\mathbf{p})^R\texttt{010110}\nu(\mathbf{p})$. Obviously $\nu(\mathbf{p})$ and $\nu(\mathbf{p})^R$ have the same set of 20 palindromes and it is easy to check that $\mathbf{x}$ contains no additional palindrome. We show that $\mathbf{x}$ is $\frac{5}{2}^+$-free by checking the central factor of $\mathbf{x}$ of length 200. Then larger repetitions of exponent $> \frac{5}{2}$ are ruled out since the word `110011001001101` is a prefix of $\texttt{110}\nu(\mathbf{p})$ but is neither a factor of $\nu(\mathbf{p})$ nor $\nu(\mathbf{p})^{\mathbf{R}}$. By symmetry, this also holds for $\mathbf{x}^R = \nu(\mathbf{p})^R\texttt{011010}\nu(\mathbf{p})$, $\overline{\mathbf{x}}$, and $\overline{\mathbf{x}^R}$.

4 The critical exponent of $\nu(\mathbf{p})$ and $\mu(\mathbf{p})$

Before recalling the definition of the infinite words $\mathbf{p}$, $\nu(\mathbf{p})$ and $\mu(\mathbf{p})$, let us underline that all of them are uniformly recurrent and $\nu(\mathbf{p})$ and $\mu(\mathbf{p})$ are morphic images of $\mathbf{p}$. Hence in order to compute their (asymptotic) critical exponents, we will exploit the following two useful statements. See also [8].

Theorem 8 ([4])**.** *Let $\mathbf{u}$ be a uniformly recurrent aperiodic infinite word. Let (w_n) be a sequence of all bispecial factors ordered by their length. For every $n \in \mathbb{N}$, let r_n be a shortest return word to w_n in $\mathbf{u}$. Then*

$$E(\mathbf{u}) = 1 + \sup_{n\in\mathbb{N}} \left\{ \frac{|w_n|}{|r_n|} \right\} \qquad \textit{and} \qquad E^*(\mathbf{u}) = 1 + \limsup_{n\to+\infty} \frac{|w_n|}{|r_n|}. \tag{1}$$

Theorem 9. *Let $\mathbf{u}$ be an infinite word over an alphabet $\mathcal{A}$ such that the uniform letter frequencies in $\mathbf{u}$ exist. Let $\psi : \mathcal{A}^* \to \mathcal{B}^*$ be an injective morphism and let $L \in \mathbb{N}$ be such that every factor v of $\psi(\mathbf{u})$, $|v| \geqslant L$, has a synchronization point. Then $E^*(\mathbf{u}) = E^*(\psi(\mathbf{u}))$.*

Proof. The inequality $E^*(\psi(\mathbf{u})) \geqslant E^*(\mathbf{u})$ is proven in [5] for any non-erasing morphism under the assumption of existence of uniform letter frequencies in $\mathbf{u}$. Let us prove the opposite inequality. According to the definition of $E^*(\psi(\mathbf{u}))$, there exist sequences (w_n) and (v_n) such that

1. $\lim_{n\to\infty} |v_n| = \infty$;
2. w_n is a factor of $\psi(\mathbf{u})$ for each $n \in \mathbb{N}$;
3. w_n is a prefix of the periodic word $(v_n)^\omega$ for each $n \in \mathbb{N}$;

4. $E^*(\psi(\mathbf{u})) = \lim_{n\to\infty} \frac{|w_n|}{|v_n|}$.

If $E^*(\psi(\mathbf{u})) = 1$, then, clearly, $E^*(\psi(\mathbf{u})) \leqslant E^*(\mathbf{u})$. Assume in the sequel that $E^*(\psi(\mathbf{u})) > 1$, then we have for large enough n that $|w_n| > |v_n|$ and moreover, by the first item, $|v_n| \geqslant L$. By assumption, both v_n and w_n have synchronization points and since v_n is a prefix of w_n for large enough n, we may write

$$w_n = x_n \bullet \psi(w'_n) \bullet y_n \quad \text{and} \quad v_n = x_n \bullet \psi(v'_n) \bullet z_n \,,$$

where we highlighted the first and the last synchronization point (not necessarily distinct) in w_n and v_n and where w'_n and v'_n are uniquely given factors of $\mathbf{u}$ and the lengths of x_n, y_n, z_n are smaller than L.

By the third item, we have

$$w_n = v_n^k u_n = (x_n \psi(v'_n) z_n)^k u_n \,,$$

where u_n is a proper prefix of v_n and $k \in \mathbb{N}, k \geqslant 1$.

There are two possible cases for (u_n).

(a) Either $(|u_n|)$ is bounded, but as $E^*(\psi(\mathbf{u})) > 1$, it follows that $k \geqslant 2$ for large enough n.

(b) Or there is a subsequence (u_{j_n}) of (u_n) such that for all $n \in \mathbb{N}$ we have $|u_{j_n}| \geqslant L$. Then by assumption, u_{j_n} has a synchronization point and we may write $u_{j_n} = x_{j_n} \bullet \psi(u'_{j_n}) \bullet y_{j_n}$, where we highlighted the first and the last synchronization point in u_{j_n} and u'_{j_n} is a prefix of v'_{j_n} by injectivity of ψ.

(a) In the first case, since $k \geqslant 2$ for large enough n, then w_n starts with $(x_n\psi(v'_n)z_n)^2$. By definition of synchronization points and injectivity of ψ, there exists a unique factor t_n of $\mathbf{u}$ such that $\psi(v'_n)z_n x_n = \psi(t_n)$. Consequently, $w_n = (x_n\psi(v'_n)z_n)^k u_n = x_n\psi(t_n^{k-1}v'_n)z_n u_n$. Therefore, $t_n^{k-1}v'_n$ is a factor of $\mathbf{u}$ and it is a prefix of $(t_n)^\omega$ and

$$E^*(\psi(\mathbf{u})) = \lim_{n\to\infty} \frac{|w_n|}{|v_n|} = \lim_{n\to\infty} \frac{|x_n\psi(t_n{}^{k-1}v'_n)z_n u_n|}{|\psi(t_n)|} = \lim_{n\to\infty} \frac{|\psi(t_n{}^{k-1}v'_n)|}{|\psi(t_n)|} \,,$$

where the last equality holds thanks to boundedness of $(|x_n|), (|z_n|)$ and $(|u_n|)$.

(b) In the second case, $w_{j_n} = (x_{j_n}\psi(v'_{j_n})z_{j_n})^k x_{j_n}\psi(u'_{j_n})y_{j_n}$, where $k \geqslant 1$. By definition of synchronization points and injectivity of ψ, there exists a unique factor t_{j_n} of $\mathbf{u}$ such that $\psi(v'_{j_n})z_{j_n}x_{j_n} = \psi(t_{j_n})$. Consequently, $(t_{j_n})^k u'_{j_n}$ is a factor of $\mathbf{u}$ and it is a prefix of $(t_{j_n})^\omega$ and

$$E^*(\psi(\mathbf{u})) = \lim_{n\to\infty} \frac{|w_n|}{|v_n|} = \lim_{n\to\infty} \frac{|x_{j_n}\psi\big((t_{j_n})^k u'_{j_n}\big)y_{j_n}|}{|\psi(t_{j_n})|} = \lim_{n\to\infty} \frac{|\psi\big((t_{j_n})^k u'_{j_n}\big)|}{|\psi(t_{j_n})|} \,,$$

where the last equality holds thanks to boundedness of (x_n) and (y_n).

Combining two simple facts:

- $\frac{|\psi(u)|}{|u|} = \vec{1}^T M_\psi \frac{\vec{u}}{|u|}$ for each word u over $\mathcal{A}$, where $\vec{1}$ is a vector with all coordinates equal to one;
- for each sequence (s_n) of factors of $\mathbf{u}$ with $\lim_{n\to\infty} |s_n| = \infty$ we have, by uniform letter frequencies in $\mathbf{u}$, $\lim\limits_{n\to\infty} \frac{\vec{s_n}}{|s_n|} = \vec{f}$, where $\vec{f}$ is the vector of letter frequencies in $\mathbf{u}$,

we obtain

$$\lim_{n\to\infty} \frac{|\psi(s_n)|}{|s_n|} = \vec{1}^T M_\psi \vec{f}. \tag{2}$$

Consequently,

(a) in the first case, since $\lim_{n\to\infty} |t_n| = \infty$, we obtain using (2)

$$\begin{aligned} E^*(\psi(\mathbf{u})) &= \lim_{n\to\infty} \frac{|\psi({t_n}^{k-1} v'_n)|}{|\psi(t_n)|} \\ &= \lim_{n\to\infty} \frac{|\psi({t_n}^{k-1} v'_n)|}{|{t_n}^{k-1} v'_n|} \frac{|t_n|}{|\psi(t_n)|} \frac{|{t_n}^{k-1} v'_n|}{|t_n|} \\ &= \lim_{n\to\infty} \frac{|{t_n}^{k-1} v'_n|}{|t_n|} \leqslant E^*(\mathbf{u}) \, , \end{aligned}$$

where the last inequality follows from the fact that $(t_n)^{k-1} v'_n \in \mathcal{L}(\mathbf{u})$ and $(t_n)^{k-1} v'_n$ is a power of t_n;

(b) in the second case, since $\lim |t_{j_n}| = \infty$, we obtain using (2)

$$\begin{aligned} E^*(\psi(\mathbf{u})) &= \lim_{n\to\infty} \frac{|\psi\big((t_{j_n})^k u'_{j_n}\big)|}{|\psi(t_{j_n})|} \\ &= \lim_{n\to\infty} \frac{|\psi\big((t_{j_n})^k u'_{j_n}\big)|}{|(t_{j_n})^k u'_{j_n}|} \frac{|t_{j_n}|}{|\psi(t_{j_n})|} \frac{|(t_{j_n})^k u'_{j_n}|}{|t_{j_n}|} \\ &= \lim_{n\to\infty} \frac{|(t_{j_n})^k u'_{j_n}|}{|t_{j_n}|} \leqslant E^*(\mathbf{u}) \, , \end{aligned}$$

where the last inequality follows from the fact that $(t_{j_n})^k u'_{j_n} \in \mathcal{L}(\mathbf{u})$ and $(t_{j_n})^k u'_{j_n}$ is a power of t_{j_n}. □

4.1 The infinite word p

In order to compute the critical exponent of morphic images of $\mathbf{p}$, it is essential to describe bispecial factors and their return words in $\mathbf{p}$ and to determine the asymptotic critical exponent of $\mathbf{p}$.

The infinite word $\mathbf{p}$ is the fixed point of the injective morphism φ, where

$$\begin{aligned} \varphi(0) &= 01, \\ \varphi(1) &= 21, \\ \varphi(2) &= 0. \end{aligned}$$

Therefore, $\mathbf{p}$ has the following prefix

$$\mathbf{p} = \mathtt{0121021010210121010210121021012101 0}\cdots$$

Remark 10. It is readily seen that each non-empty factor of $\mathbf{p}$ has a synchronization point.

The following characteristics of $\mathbf{p}$ are known [2]:

- The factor complexity of $\mathbf{p}$ is $C(n) = 2n + 1$.
- The word $\mathbf{p}$ is not closed under reversal: $\mathtt{02} \in \mathcal{L}(\mathbf{p})$, but $\mathtt{20} \notin \mathcal{L}(\mathbf{p})$.
- The word $\mathbf{p}$ is uniformly recurrent and $\mathbf{p}$ has uniform letter frequencies because φ is primitive.

4.1.1 Bispecial factors in p

First, we will examine LS factors. Using the form of φ, we observe

- $\mathtt{0}$ has only one left extension: $\mathtt{1}$,
- $\mathtt{1}$ has two left extensions: $\mathtt{0}$ and $\mathtt{2}$,
- $\mathtt{2}$ has two left extensions: $\mathtt{0}$ and $\mathtt{1}$.

Therefore, every LS factor has left extensions either $\{\mathtt{0}, \mathtt{2}\}$, or $\{\mathtt{0}, \mathtt{1}\}$.

Lemma 11. *Let $w \neq \varepsilon$, $w \in \mathcal{L}(\mathbf{p})$.*

- *If w is a LS factor such that $\mathtt{0}w, \mathtt{1}w \in \mathcal{L}(\mathbf{p})$, then $\mathtt{1}\varphi(w)$ is a LS factor such that $\mathtt{01}\varphi(w), \mathtt{21}\varphi(w) \in \mathcal{L}(\mathbf{p})$.*
- *If w is a LS factor such that $\mathtt{0}w, \mathtt{2}w \in \mathcal{L}(\mathbf{p})$, then $\varphi(w)$ is a LS factor such that $\mathtt{0}\varphi(w), \mathtt{1}\varphi(w) \in \mathcal{L}(\mathbf{p})$.*

Proof. It follows from the form of φ and the fact that $\mathbf{p}$ is the fixed point of the morphism φ, i.e., if $u \in \mathcal{L}(\mathbf{p})$, then $\varphi(u) \in \mathcal{L}(\mathbf{p})$. □

Second, we will focus on RS factors. We observe

- $\mathtt{0}$ has two right extensions: $\mathtt{1}$ and $\mathtt{2}$,
- $\mathtt{1}$ has two right extensions: $\mathtt{0}$ and $\mathtt{2}$,
- $\mathtt{2}$ has only one right extension: $\mathtt{1}$.

Therefore, every RS factor has right extensions either $\{\mathtt{1}, \mathtt{2}\}$, or $\{\mathtt{0}, \mathtt{2}\}$. Using similar arguments as for LS factors, we get the following statement.

Lemma 12. *Let $w \neq \varepsilon$, $w \in \mathcal{L}(\mathbf{p})$.*

- *If w is a RS factor such that $w\mathtt{0}, w\mathtt{2} \in \mathcal{L}(\mathbf{p})$, then $\varphi(w)\mathtt{0}$ is a RS factor such that $\varphi(w)\mathtt{01}, \varphi(w)\mathtt{02} \in \mathcal{L}(\mathbf{p})$.*
- *If w is a RS factor such that $w\mathtt{1}, w\mathtt{2} \in \mathcal{L}(\mathbf{p})$, then $\varphi(w)$ is a RS factor such that $\varphi(w)\mathtt{2}, \varphi(w)\mathtt{0} \in \mathcal{L}(\mathbf{p})$.*

It follows from the form of LS and RS factors that we have at most 4 possible kinds of non-empty BS factors in $\mathbf{p}$.

Proposition 13. *Let v be a non-empty BS factor in $\mathbf{p}$.*

1. *$\mathtt{0}v, \mathtt{2}v, v\mathtt{0}, v\mathtt{2} \in \mathcal{L}(\mathbf{p})$ if and only if there exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \mathtt{1}\varphi(w)$ and $\mathtt{0}w, \mathtt{1}w, w\mathtt{1}, w\mathtt{2} \in \mathcal{L}(\mathbf{p})$.*
2. *$\mathtt{0}v, \mathtt{1}v, v\mathtt{1}, v\mathtt{2} \in \mathcal{L}(\mathbf{p})$ if and only if there exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \varphi(w)\mathtt{0}$ and $\mathtt{0}w, \mathtt{2}w, w\mathtt{0}, w\mathtt{2} \in \mathcal{L}(\mathbf{p})$.*
3. *$\mathtt{0}v, \mathtt{2}v, v\mathtt{1}, v\mathtt{2} \in \mathcal{L}(\mathbf{p})$ if and only if there exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \mathtt{1}\varphi(w)\mathtt{0}$ and $\mathtt{0}w, \mathtt{1}w, w\mathtt{0}, w\mathtt{2} \in \mathcal{L}(\mathbf{p})$.*
4. *$\mathtt{0}v, \mathtt{1}v, v\mathtt{0}, v\mathtt{2} \in \mathcal{L}(\mathbf{p})$ if and only if there exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \varphi(w)$ and $\mathtt{0}w, \mathtt{2}w, w\mathtt{1}, w\mathtt{2} \in \mathcal{L}(\mathbf{p})$.*

Proof. The implication ($\Leftarrow$) follows from Lemmata 11 and 12. We will prove the opposite implication for Item 1, the other cases may be proven analogously. If v is a non-empty factor such that $\mathtt{0}v, \mathtt{2}v, v\mathtt{0}, v\mathtt{2} \in \mathcal{L}(\mathbf{p})$, then v necessarily starts and ends with the letter $\mathtt{1}$. By the form of φ, we have the following synchronization points $v = \mathtt{1} \bullet \hat{v} \bullet$ ($\hat{v}$ may be empty). Hence, by injectivity of φ, there exists a unique w in $\mathbf{p}$ such that $v = \mathtt{1}\varphi(w)$. Thus, using again the form of φ and the knowledge of possible right extensions, the factor w is BS and $\mathtt{0}w, \mathtt{1}w, w\mathtt{1}, w\mathtt{2} \in \mathcal{L}(\mathbf{p})$. □

We can see that the only BS factor of length one is $\mathtt{1}$, it has left extensions $\mathtt{0}, \mathtt{2}$ and right extensions $\mathtt{0}, \mathtt{2}$. Applying Proposition 13 Item 2, we obtain that $\varphi(\mathtt{1})\mathtt{0}$ is BS with left extensions $\mathtt{0}, \mathtt{1}$ and right extensions $\mathtt{1}, \mathtt{2}$. Proposition 13 Item 1 gives us that $\mathtt{1}\varphi^2(\mathtt{1})\varphi(\mathtt{0})$ is BS with left extensions $\mathtt{0}, \mathtt{2}$ and right extensions $\mathtt{0}, \mathtt{2}$. This process can be iterated providing us with infinitely many BS factors:

$$\begin{aligned} &\mathtt{1} \to \varphi(\mathtt{1})\mathtt{0} \to \mathtt{1}\varphi^2(\mathtt{1})\varphi(\mathtt{0}) \to \varphi(\mathtt{1})\varphi^3(\mathtt{1})\varphi^2(\mathtt{0})\mathtt{0} \to \\ &\to \mathtt{1}\varphi^2(\mathtt{1})\varphi^4(\mathtt{1})\varphi^3(\mathtt{0})\varphi(\mathtt{0}) \to \varphi(\mathtt{1})\varphi^3(\mathtt{1})\varphi^5(\mathtt{1})\varphi^4(\mathtt{0})\varphi^2(\mathtt{0})\mathtt{0} \quad \cdots \end{aligned} \tag{3}$$

The only BS factor of length two is $\mathtt{10}$, it has left extensions $\mathtt{0}, \mathtt{2}$ and right extensions $\mathtt{1}, \mathtt{2}$. Applying Proposition 13 Item 4, we obtain that $\varphi(\mathtt{1})\varphi(\mathtt{0})$ is BS with left extensions $\mathtt{0}, \mathtt{1}$ and right extensions $\mathtt{0}, \mathtt{2}$. Proposition 13 Item 3 gives us that $\mathtt{1}\varphi^2(\mathtt{1})\varphi^2(\mathtt{0})\mathtt{0}$ is BS with left extensions $\mathtt{0}, \mathtt{2}$ and right extensions $\mathtt{1}, \mathtt{2}$. This process can be iterated providing us again with infinitely many BS factors:

$$\begin{aligned} &\mathtt{10} \to \varphi(\mathtt{1})\varphi(\mathtt{0}) \to \mathtt{1}\varphi^2(\mathtt{1})\varphi^2(\mathtt{0})\mathtt{0} \to \varphi(\mathtt{1})\varphi^3(\mathtt{1})\varphi^3(\mathtt{0})\varphi(\mathtt{0}) \to \\ &\to \mathtt{1}\varphi^2(\mathtt{1})\varphi^4(\mathtt{1})\varphi^4(\mathtt{0})\varphi^2(\mathtt{0})\mathtt{0} \to \varphi(\mathtt{1})\varphi^3(\mathtt{1})\varphi^5(\mathtt{1})\varphi^5(\mathtt{0})\varphi^3(\mathtt{0})\varphi(\mathtt{0}) \quad \cdots \end{aligned} \tag{4}$$

Each BS factor v of length greater than two has at least two synchronization points and the corresponding BS factor w from Proposition 13 is non-empty. In other words, the BS factor v makes part of one of the sequences (3) and (4) of BS factors.

As a consequence of Proposition 13 and the above arguments, we get a complete description of BS factors in $\mathbf{p}$.

Corollary 14. *Let w be a non-empty BS factor in $\mathbf{p}$. Then it has one of the following forms:*

($\mathcal{A}$)
$$w_A^{(n)} = \mathtt{1}\varphi^2(\mathtt{1})\varphi^4(\mathtt{1})\cdots\varphi^{2n}(\mathtt{1})\varphi^{2n-1}(\mathtt{0})\varphi^{2n-3}(\mathtt{0})\cdots\varphi(\mathtt{0})$$
for $n \geqslant 1$. If $n = 0$, then we set $w_A^{(0)} = \mathtt{1}$.

The Parikh vector of $w_A^{(n)}$ is the same as of the word $\mathtt{1}\varphi(\mathtt{012})\varphi^3(\mathtt{012})\cdots\varphi^{2n-1}(\mathtt{012})$.

($\mathcal{B}$)
$$w_B^{(n)} = \varphi(\mathtt{1})\varphi^3(\mathtt{1})\cdots\varphi^{2n+1}(\mathtt{1})\varphi^{2n}(\mathtt{0})\varphi^{2n-2}(\mathtt{0})\cdots\varphi^2(\mathtt{0})\mathtt{0}$$
for $n \geqslant 0$.

The Parikh vector of $w_B^{(n)}$ is the same as of the word $\mathtt{012}\varphi^2(\mathtt{012})\varphi^4(\mathtt{012})\cdots\varphi^{2n}(\mathtt{012})$.

($\mathcal{C}$)
$$w_C^{(n)} = \mathtt{1}\varphi^2(\mathtt{1})\varphi^4(\mathtt{1})\cdots\varphi^{2n}(\mathtt{1})\varphi^{2n}(\mathtt{0})\varphi^{2n-2}(\mathtt{0})\cdots\varphi^2(\mathtt{0})\mathtt{0}$$
for $n \geqslant 0$.

The Parikh vector of $w_C^{(n)}$ is the same as of the word $\mathtt{01}\varphi^2(\mathtt{01})\varphi^4(\mathtt{01})\cdots\varphi^{2n}(\mathtt{01})$.

($\mathcal{D}$)
$$w_D^{(n)} = \varphi(\mathtt{1})\varphi^3(\mathtt{1})\cdots\varphi^{2n+1}(\mathtt{1})\varphi^{2n+1}(\mathtt{0})\varphi^{2n-1}(\mathtt{0})\cdots\varphi(\mathtt{0})$$
for $n \geqslant 0$.

The Parikh vector of $w_D^{(n)}$ is the same as of the word $\varphi(\mathtt{01})\varphi^3(\mathtt{01})\cdots\varphi^{2n+1}(\mathtt{01})$.

Lemma 15. *All BS factors in $\mathbf{p}$ are ordinary.*

Proof. The empty word is ordinary because all factors of length two are $\mathtt{10}, \mathtt{01}, \mathtt{02}, \mathtt{12}, \mathtt{21}$. Thus $b(\varepsilon) = 5 - 3 - 3 + 1 = 0$. It is easy to verify that each non-empty BS factor w has three extensions. In particular,

- extensions of $w = w_A^{(n)}$ are: $\mathtt{0}w\mathtt{2}, \mathtt{2}w\mathtt{0}, \mathtt{0}w\mathtt{0}$,
- extensions of $w = w_B^{(n)}$ are: $\mathtt{2}w\mathtt{2}, \mathtt{2}w\mathtt{1}, \mathtt{0}w\mathtt{2}$,
- extensions of $w = w_C^{(n)}$ are: $\mathtt{0}w\mathtt{0}, \mathtt{0}w\mathtt{2}, \mathtt{1}w\mathtt{0}$,
- extensions of $w = w_D^{(n)}$ are: $\mathtt{1}w\mathtt{2}, \mathtt{0}w\mathtt{1}, \mathtt{1}w\mathtt{1}$.

Consequently, $b(w) = 3 - 2 - 2 + 1 = 0$. □

4.1.2 The shortest return words to bispecial factors in $\mathbf{p}$

Each factor of $\mathbf{p}$ has 3 return words. This claim follows from the next theorem.

Theorem 16 (Theorem 5.7 in [1])**.** *Let $\mathbf{u}$ be a uniformly recurrent infinite word. Then each factor of $\mathbf{u}$ has exactly 3 return words if and only if $C(n) = 2n + 1$ and $\mathbf{u}$ has no weak BS factors.*

Let us first comment on return words to the shortest BS factors – observe the prefix of $\mathbf{p}$ at the beginning of this section.

- The return words to ε are $\mathtt{0}, \mathtt{1}, \mathtt{2}$.
- The return words to $\mathtt{1}$ are $\mathtt{12}, \mathtt{102}, \mathtt{10}$.
- The return words to $\varphi(\mathtt{1})\mathtt{0}$ are $\mathtt{210} = \varphi(\mathtt{1})\mathtt{0}$, $\mathtt{21010}$, $\mathtt{2101}$. The shortest one is $\mathtt{210}$ and it is a prefix of all of them.
- The return words to $\mathtt{10}$ are $\mathtt{10}$, $\mathtt{102}$, $\mathtt{1012}$. The shortest one is $\mathtt{10}$ and it is a prefix of all of them.

Lemma 17. *If w is a non-empty BS factor of $\mathbf{p}$ and v is a return word to w, then $\varphi(v)$ is a return word to $\varphi(w)$.*

Proof. On one hand, since vw contains w as a prefix and as a suffix, $\varphi(v)\varphi(w)$ contains $\varphi(w)$ as a prefix and as a suffix, too. On the other hand, w starts in $\mathtt{1}$ or $\mathtt{2}$ and ends in $\mathtt{0}$ or $\mathtt{1}$, thus $\varphi(w)$ starts in $\mathtt{0}$ or $\mathtt{2}$ and ends in $\mathtt{1}$, therefore it has the following synchronization points $\bullet\varphi(w)\bullet$. Consequently, $\varphi(v)\varphi(w)$ cannot contain $\varphi(w)$ somewhere in the middle because in such a case, by injectivity of φ, vw would contain w also somewhere in the middle. ☐

The following observation is an immediate consequence of the definition of return words.

Observation 18. *Let w be a factor of $\mathbf{p}$ and let v be its return word. If w has a unique right extension a, then v is a return word to wa, too. If w has a unique left extension b, then bvb^{-1} is a return word to bw. In particular, the Parikh vectors of the corresponding return words are the same.*

Example 19. Consider the BS factor $\mathtt{10}$ with the shortest return word $\mathtt{10}$ (being a prefix of the other two return words), then by Lemma 17 the BS factor $\varphi(\mathtt{1})\varphi(\mathtt{0})$ has the shortest return word equal to $\varphi(\mathtt{10})$. By Lemma 17, the factor $\varphi^2(\mathtt{1})\varphi^2(\mathtt{0})$ has the shortest return word equal to $\varphi^2(\mathtt{10})$ and by Observation 18, the shortest return word to the BS factor $\mathtt{1}\varphi^2(\mathtt{1})\varphi^2(\mathtt{0})\mathtt{0}$ has the same Parikh vector as $\varphi^2(\mathtt{10})$.

Putting together Lemma 17, Observation 18 and the knowledge of BS factors, we obtain the following statement about the shortest return words to BS factors in $\mathbf{p}$.

Corollary 20. *The shortest return words to BS factors in $\mathbf{p}$ have the following properties.*

($\mathcal{A}$) *The shortest return words to* $w_A^{(n)}$ *are*

- *(i)* $\mathtt{12}$ *and* $\mathtt{10}$ *for* $n = 0$,
- *(ii)* $r_A^{(n)}$ *with the same Parikh vector as* $\varphi^{2n-1}(\mathtt{012})$ *for* $n \geqslant 1$.

($\mathcal{B}$) *The shortest return word* $r_B^{(n)}$ *to* $w_B^{(n)}$ *has the same Parikh vector as* $\varphi^{2n}(\mathtt{012})$.

($\mathcal{C}$) *The shortest return word to* $w_C^{(n)}$ *is*

- *(i)* $\mathtt{10}$ *for* $n = 0$
- *(ii)* $r_C^{(n)}$ *with the same Parikh vector as* $\varphi^{2n}(\mathtt{01})$ *for* $n \geqslant 1$.

($\mathcal{D}$) *The shortest return word* $r_D^{(n)}$ *to* $w_D^{(n)}$ *has the same Parikh vector as* $\varphi^{2n+1}(\mathtt{01})$.

Proof. We will prove case ($\mathcal{A}$). The other cases are similar. The shortest return words to $w_A^{(0)} = \mathtt{1}$ are given at the beginning of Section 4.1.2. Let us proceed by induction on n. Consider the bispecial factor $w_A^{(1)} = \mathtt{1}\varphi^2(\mathtt{1})\varphi(\mathtt{0}) = \mathtt{1}\varphi(\varphi(\mathtt{1})\mathtt{0})$. By description of the shortest return words to short bispecial factors, we know that $\mathtt{210}$ is the shortest return word (moreover prefix of all other return words) to the bispecial factor $\varphi(\mathtt{1})\mathtt{0}$. Using Lemma 17, $\varphi(\mathtt{210})$ is the shortest return word to the factor $\varphi^2(\mathtt{1})\varphi(\mathtt{0})$. By Observation 18, the Parikh vector of the shortest return word to $w_A^{(1)} = \mathtt{1}\varphi^2(\mathtt{1})\varphi(\mathtt{0})$ is equal to the Parikh vector of $\varphi(\mathtt{210})$, hence also to the Parikh vector of $\varphi(\mathtt{012})$. Assume for a fixed $n \geqslant 1$, the bispecial factor $w_A^{(n)} = \mathtt{1}\varphi^2(\mathtt{1})\varphi^4(\mathtt{1})\cdots\varphi^{2n}(\mathtt{1})\varphi^{2n-1}(\mathtt{0})\varphi^{2n-3}(\mathtt{0})\cdots\varphi(\mathtt{0})$ has the shortest return word with the Parikh vector $\varphi^{2n-1}(\mathtt{012})$ and this return word is a prefix of all other return words. By definition, $w_A^{(n+1)} = \mathtt{1}\varphi^2(w_A^{(n)})\varphi(\mathtt{0})$. By Lemma 17 and by induction assumption, the shortest return word to the factor $\varphi^2(w_A^{(n)})$ has the same Parikh vector as $\varphi^{2n+1}(\mathtt{012})$. Using Observation 18, we obtain that the shortest return word to $w_A^{(n+1)}$ has the same Parikh vector as the factor $\varphi^{2n+1}(\mathtt{012})$, too. □

4.1.3 The asymptotic critical exponent of p

Let us determine the asymptotic critical exponent of $\mathbf{p}$ using Theorem 8. We use the form of BS factors and their shortest return words determined above. We get $E^*(\mathbf{p}) = 1 + \max\{A', B', C', D'\}$, where

$$A' = \limsup_{n\to\infty} \frac{|w_A^{(n)}|}{|r_A^{(n)}|} = \limsup_{n\to\infty} \frac{|\mathtt{1}\varphi(\mathtt{012})\varphi^3(\mathtt{012})\cdots\varphi^{2n-1}(\mathtt{012}))|}{|\varphi^{2n-1}(\mathtt{012})|};$$

$$B' = \limsup_{n\to\infty} \frac{|w_B^{(n)}|}{|r_B^{(n)}|} = \limsup_{n\to\infty} \frac{|\mathtt{012}\varphi^2(\mathtt{012})\varphi^4(\mathtt{012})\cdots\varphi^{2n}(\mathtt{012}))|}{|\varphi^{2n}(\mathtt{012})|};$$

$$C' = \limsup_{n\to\infty} \frac{|w_C^{(n)}|}{|r_C^{(n)}|} = \limsup_{n\to\infty} \frac{|\mathtt{01}\varphi^2(\mathtt{01})\varphi^4(\mathtt{01})\cdots\varphi^{2n}(\mathtt{01})|}{|\varphi^{2n}(\mathtt{01})|};$$

$$D' = \limsup_{n\to\infty} \frac{|w_D^{(n)}|}{|r_D^{(n)}|} = \limsup_{n\to\infty} \frac{|\varphi(\mathtt{01})\varphi^3(\mathtt{01})\cdots\varphi^{2n+1}(\mathtt{01})|}{|\varphi^{2n+1}(\mathtt{01})|}.$$

By the Hamilton-Cayley theorem, we have $M_\varphi^3 - 2M_\varphi^2 + M_\varphi - I = 0$. Consequently, for each $w \in \{0,1,2\}^*$, if we denote $\ell_n := |\varphi^n(w)| = (1,1,1)M_\varphi^n \vec{w}$, then ℓ_n satisfies the recurrence relation $\ell_{n+3} - 2\ell_{n+2} + \ell_{n+1} - \ell_n = 0$. Denote β the largest root of the characteristic polynomial $t^3 - 2t^2 + t - 1$; $\beta \doteq 1.75488$. By the Perron-Frobenius theorem, β is strictly larger than the modulus of the other roots of the characteristic polynomial. We thus obtain:

$$\begin{aligned} A' &= \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \beta^{2k-1}}{\beta^{2n-1}} = \tfrac{\beta^2}{\beta^2-1}; \\ B' = C' &= \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \beta^{2k}}{\beta^{2n}} = \tfrac{\beta^2}{\beta^2-1}; \\ D' &= \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \beta^{2k+1}}{\beta^{2n+1}} = \tfrac{\beta^2}{\beta^2-1}. \end{aligned}$$

Consequently, $E^*(\mathbf{p}) = 1 + \frac{\beta^2}{\beta^2-1} \doteq 2.48$.

4.2 The infinite word $\boldsymbol{\nu}(\mathbf{p})$

The morphism ν has the form:

$$\begin{aligned} \nu(0) &= 011, \\ \nu(1) &= 0, \\ \nu(2) &= 01. \end{aligned}$$

Therefore,

$$\nu(\mathbf{p}) = 0110010011010011001101001100100 11\cdots$$

and ν is injective.

Remark 21. The reader may easily check that any factor of $\nu(\mathbf{p})$ of length at least two has a synchronization point.

Using the above remark and Theorem 9, we deduce that

$$E^*(\nu(\mathbf{p})) = E^*(\mathbf{p}).$$

4.2.1 Bispecial factors in $\boldsymbol{\nu}(\mathbf{p})$

Lemma 22. *Let $v \in \mathcal{L}(\nu(\mathbf{p}))$ be a BS factor of length at least two. Then one of the items holds.*

1. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = 1\nu(w)01$ and $0w, 2w, w0, w2 \in \mathcal{L}(\mathbf{p})$.*
2. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \nu(w)0$ and $0w, 1w, w1, w2 \in \mathcal{L}(\mathbf{p})$.*
3. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = 1\nu(w)0$ and $0w, 2w, w1, w2 \in \mathcal{L}(\mathbf{p})$.*

4. *There exists* $w \in \mathcal{L}(\mathbf{p})$ *such that* $v = \nu(w)\mathtt{01}$ *and* $\mathtt{0}w, \mathtt{1}w, w\mathtt{0}, w\mathtt{2} \in \mathcal{L}(\mathbf{p})$.

Proof. The statement follows from Remark 21 and from the possible left and right extensions of factors in $\mathbf{p}$. □

Combining Lemma 22 and Corollary 14, we get a complete description of BS factors in $\nu(\mathbf{p})$.

Corollary 23. *Let* v *be a non-empty BS factor in* $\nu(\mathbf{p})$ *of length at least two. Then* $v = \mathtt{01}$ *or* $v = \mathtt{10}$ *or* v *has one of the following forms:*

($\mathcal{A}$)

$$v_A^{(n)} = \mathtt{1}\nu(\mathtt{1}\varphi^2(\mathtt{1})\varphi^4(\mathtt{1})\cdots\varphi^{2n}(\mathtt{1})\varphi^{2n-1}(\mathtt{0})\varphi^{2n-3}(\mathtt{0})\cdots\varphi(\mathtt{0}))\mathtt{01}$$

for $n \geqslant 1$ *and* $v_A^{(0)} = \mathtt{1}\nu(\mathtt{1})\mathtt{01} = \mathtt{1001}$.

$v_A^{(n)}$ *and* $\mathtt{011}\nu(\mathtt{1}\varphi(\mathtt{012})\varphi^3(\mathtt{012})\ldots\varphi^{2n-1}(\mathtt{012}))$ *have the same Parikh vector.*

($\mathcal{B}$)

$$v_B^{(n)} = \nu(\varphi(\mathtt{1})\varphi^3(\mathtt{1})\cdots\varphi^{2n+1}(\mathtt{1})\varphi^{2n}(\mathtt{0})\varphi^{2n-2}(\mathtt{0})\cdots\varphi^2(\mathtt{0})\mathtt{0})\mathtt{0}$$

for $n \geqslant 0$.

$v_B^{(n)}$ *and* $\mathtt{0}\nu(\mathtt{012}\varphi^2(\mathtt{012})\varphi^4(\mathtt{012})\ldots\varphi^{2n}(\mathtt{012}))$ *have the same Parikh vector.*

($\mathcal{C}$)

$$v_C^{(n)} = \mathtt{1}\nu(\mathtt{1}\varphi^2(\mathtt{1})\varphi^4(\mathtt{1})\cdots\varphi^{2n}(\mathtt{1})\varphi^{2n}(\mathtt{0})\varphi^{2n-2}(\mathtt{0})\cdots\varphi^2(\mathtt{0})\mathtt{0})\mathtt{0}$$

for $n \geqslant 0$.

$v_C^{(n)}$ *and* $\mathtt{01}\nu(\mathtt{01}\varphi^2(\mathtt{01})\varphi^4(\mathtt{01})\ldots\varphi^{2n}(\mathtt{01}))$ *have the same Parikh vector.*

($\mathcal{D}$)

$$v_D^{(n)} = \nu(\varphi(\mathtt{1})\varphi^3(\mathtt{1})\cdots\varphi^{2n+1}(\mathtt{1})\varphi^{2n+1}(\mathtt{0})\varphi^{2n-1}(\mathtt{0})\cdots\varphi(\mathtt{0}))\mathtt{01}$$

for $n \geqslant 0$.

$v_D^{(n)}$ *and* $\mathtt{01}\nu(\varphi(\mathtt{01})\varphi^3(\mathtt{01})\cdots\varphi^{2n+1}(\mathtt{01}))$ *have the same Parikh vector.*

4.2.2 The shortest return words to bispecial factors in $\nu(\mathbf{p})$

Lemma 24. *If* w *is a non-empty BS factor in* $\mathbf{p}$ *and* v *is its return word, then* $\nu(v)$ *is a return word to* $\nu(w)\mathtt{0}$.

Proof. On one hand, consider any occurrence of vw and denote a the following letter, then $\nu(v)\nu(w)\mathtt{0}$ is a prefix of $\nu(vwa)$. Since vw contains w as a prefix and as a suffix, then $\nu(v)\nu(w)\mathtt{0}$ contains $\nu(w)\mathtt{0}$ as a prefix and as a suffix, too. On the other hand, w starts in $\mathtt{1}$ or $\mathtt{2}$ and ends in $\mathtt{0}$ or $\mathtt{1}$, thus $\nu(w)\mathtt{0}$ starts in $\mathtt{0}$ and ends in $\mathtt{0110}$ or $\mathtt{00}$, therefore $\nu(w)\mathtt{0}$ has the following synchronization points $\bullet\nu(w)\bullet\mathtt{0}$. Consequently, $\nu(v)\nu(w)\mathtt{0}$ cannot contain $\nu(w)\mathtt{0}$ somewhere in the middle because in such a case, by injectivity of ν, vw would contain w also somewhere in the middle. □

Applying Lemma 24 and Observation 18, we have the following description of the shortest return words to BS factors.

Corollary 25. *The shortest return words to BS factors of length at least three in* $\nu(\mathbf{p})$ *have the following properties.*

($\mathcal{A}$) *The shortest return word* $\hat{r}_A^{(n)}$ *to* $v_A^{(n)}$ *has the same Parikh vector as* $\nu(\varphi^{2n-1}(\mathtt{012}))$ *for* $n \geqslant 1$ *and* $\mathtt{100}$ *is the shortest return word to* $v_A^{(0)} = \mathtt{1001}$.

($\mathcal{B}$) *The shortest return word* $\hat{r}_B^{(n)}$ *to* $v_B^{(n)}$ *has the same Parikh vector as* $\nu(\varphi^{2n}(\mathtt{012}))$.

($\mathcal{C}$) *The shortest return word* $\hat{r}_C^{(n)}$ *to* $v_C^{(n)}$ *has the same Parikh vector as* $\nu(\varphi^{2n}(\mathtt{01}))$.

($\mathcal{D}$) *The shortest return word* $\hat{r}_D^{(n)}$ *to* $v_D^{(n)}$ *has the same Parikh vector as* $\nu(\varphi^{2n+1}(\mathtt{01}))$.

Proof. We will prove case ($\mathcal{A}$). The other cases are similar. We know that the return words to $w_A^{(0)} = \mathtt{1}$ in $\mathbf{p}$ are $\mathtt{12}, \mathtt{10}, \mathtt{102}$. Using Lemma 24, we obtain that $\nu(\mathtt{12}) = \mathtt{001}$, $\nu(\mathtt{10}) = \mathtt{0011}$ and $\nu(\mathtt{102}) = \mathtt{001101}$ are return words to $\nu(\mathtt{1})0$. Since $\nu(\mathtt{1})0$ has unique left and right extensions $\mathtt{1}$, using twice Observation 18, we obtain that $\mathtt{1}\nu(\mathtt{12})\mathtt{1}^{-1} = \mathtt{100}$, $\mathtt{1001}$ and $\mathtt{100110}$ are return words to $v_A^0 = \mathtt{1001}$. Therefore, the shortest return word to $v_A^{(0)} = \mathtt{1001}$ is $\mathtt{100}$ and it is a prefix of all of them.

Now, let us consider $n \geqslant 1$ and the bispecial factor

$$v_A^{(n)} = \mathtt{1}\nu(\mathtt{1}\varphi^2(\mathtt{1})\varphi^4(\mathtt{1})\cdots\varphi^{2n}(\mathtt{1})\varphi^{2n-1}(\mathtt{0})\varphi^{2n-3}(\mathtt{0})\cdots\varphi(\mathtt{0}))\mathtt{01} = \mathtt{1}\nu(w_A^{(n)})\mathtt{01}.$$

Using Corollary 20, we know that the shortest return word to $w_A^{(n)}$ has the same Parikh vector as $\varphi^{2n-1}(\mathtt{012})$, moreover the shortest return word is a prefix of all other return words.

Using Lemma 24, and the fact that ν is non-erasing, we obtain that the shortest return word to $\nu(w_A^{(n)})\mathtt{0}$ has the same Parikh vector as $\nu(\varphi^{2n-1}(\mathtt{012}))$. Using Observation 18 twice, we obtain that the shortest return word to $\mathtt{1}\nu(w_A^{(n)})\mathtt{01}$ has the same Parikh vector as $\nu(\varphi^{2n-1}(\mathtt{012}))$, since adding $\mathtt{1}$ at the beginning and erasing $\mathtt{1}$ at the end does not change the Parikh vector. □

4.2.3 The critical exponent of $\nu(\mathbf{p})$

Using Theorem 8 and the description of BS factors from Corollary 23 and of their shortest return words from Corollary 25, we obtain the following formula for the critical exponent of $\nu(\mathbf{p})$.

$$E(\nu(\mathbf{p})) = 1 + \max\{A, B, C, D, F\} \ ,$$

where

$$A = \sup_{n\geqslant 1}\left\{\frac{|v_A^{(n)}|}{|\hat{r}_A^{(n)}|}\right\} = \sup_{n\geqslant 1}\left\{\frac{|011\nu(1\varphi(012)\varphi^3(012)\ldots\varphi^{2n-1}(012))|}{|\nu(\varphi^{2n-1}(012))|}\right\} \cup \left\{\frac{|1001|}{|100|}\right\};$$

$$B = \sup_{n\geqslant 0}\left\{\frac{|v_B^{(n)}|}{|\hat{r}_B^{(n)}|}\right\} = \sup_{n\geqslant 0}\left\{\frac{|0\nu(012\varphi^2(012)\varphi^4(012)\ldots\varphi^{2n}(012))|}{|\nu(\varphi^{2n}(012))|}\right\};$$

$$C = \sup_{n\geqslant 0}\left\{\frac{|v_C^{(n)}|}{|\hat{r}_C^{(n)}|}\right\} = \sup_{n\geqslant 1}\left\{\frac{|01\nu(01\varphi^2(01)\varphi^4(01)\ldots\varphi^{2n}(01))|}{|\nu(\varphi^{2n}(01))|}\right\} \cup \left\{\frac{|1\nu(10)0|}{|\nu(10)|}\right\};$$

$$D = \sup_{n\geqslant 0}\left\{\frac{|v_D^{(n)}|}{|\hat{r}_D^{(n)}|}\right\} = \sup_{n\geqslant 0}\left\{\frac{|01\nu(\varphi(01)\varphi^3(01)\ldots\varphi^{2n+1}(01))|}{|\nu(\varphi^{2n+1}(01))|}\right\};$$

$$F = \max\left\{\frac{|w|}{|r|} : w \text{ BS in } \nu(\mathbf{p}) \text{ of length one or two and } r \text{ its shortest return word}\right\}.$$

Theorem 26. *The critical exponent of $\nu(\mathbf{p})$ equals*

$$E(\nu(\mathbf{p})) = \frac{5}{2}.$$

Proof. To evaluate the critical exponent of $\nu(\mathbf{p})$ using the above formula, we have to do several steps.

1. Determining the shortest return words of BS factors of length one and two in $\nu(\mathbf{p})$:

   - 0 is a BS factor with the shortest return word 0.
   - 1 is a BS factor with the shortest return word 1.
   - 01 is a BS factor with the shortest return words 010, 011.
   - 10 is a BS factor with the shortest return word 10.

   Thus for each BS factor w of length one or two and its shortest return word r we have $\frac{|w|}{|r|} \leqslant 1$ and $F = 1$.

2. Computation of A and B. The sequence $c_n := |\nu(\varphi^n(012))|$ satisfies $c_0 = 6, c_1 = 10, c_2 = 17$, and the recurrence relation $c_n = 2c_{n-1} - c_{n-2} + c_{n-3}$.

   The explicit solution reads

   $$c_n = A_1\beta^n + B_1\lambda_1^n + C_1\lambda_2^n,$$

   where

   $$\beta \doteq 1.75488, \quad \lambda_1 \doteq 0.12256 + 0.74486i, \quad \lambda_2 = \overline{\lambda_1}$$

are the roots of the polynomial $t^3 - 2t^2 + t - 1$, and

$$\begin{aligned} A_1 &= \frac{6|\lambda_1|^2 - 20\,\mathrm{Re}(\lambda_1) + 17}{|\beta - \lambda_1|^2} \doteq 5.581308964\,; \\ B_1 &= \frac{6\beta\lambda_2 - 10(\beta + \lambda_2) + 17}{(\beta - \lambda_1)(\lambda_2 - \lambda_1)} \doteq 0.209345518 - 0.103481025i\,; \\ C_1 &= \overline{B_1}\,. \end{aligned}$$

Let us show that $A \leqslant \frac{3}{2}$. Since $\frac{|\mathtt{1001}|}{|\mathtt{100}|} = \frac{4}{3} < \frac{3}{2}$, it remains to show for all $n \geqslant 1$ that

$$\begin{aligned} \frac{4 + A_1 \sum_{k=1}^{n} \beta^{2k-1} + B_1 \sum_{k=1}^{n} \lambda_1^{2k-1} + C_1 \sum_{k=1}^{n} \lambda_2^{2k-1}}{A_1 \beta^{2n-1} + B_1 \lambda_1^{2n-1} + C_1 \lambda_2^{2n-1}} &\overset{?}{\leqslant} \frac{3}{2}, \\ 8 + 2A_1 \sum_{k=1}^{n} \beta^{2k-1} + 4\,\mathrm{Re}\left(B_1 \sum_{k=1}^{n} \lambda_1^{2k-1}\right) &\overset{?}{\leqslant} 3A_1\beta^{2n-1} + 6\,\mathrm{Re}(B_1\lambda_1^{2n-1}), \\ 8 + 2A_1 \sum_{k=1}^{n-1} \beta^{2k-1} + 4\,\mathrm{Re}\left(B_1 \sum_{k=1}^{n-1} \lambda_1^{2k-1}\right) &\overset{?}{\leqslant} A_1\beta^{2n-1} + 2\,\mathrm{Re}(B_1\lambda_1^{2n-1}), \\ 8 + 2A_1\left(\frac{\beta^{2n-1}}{\beta^2 - 1} - \frac{\beta}{\beta^2 - 1}\right) + 4\,\mathrm{Re}\left(B_1\lambda_1 \frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) &\overset{?}{\leqslant} A_1\beta^{2n-1} + 2\,\mathrm{Re}(B_1\lambda_1^{2n-1}). \end{aligned}$$

Since

$$\frac{2}{\beta^2 - 1} \leqslant 1,$$

we need to prove the inequality in the form

$$8 + 4\,\mathrm{Re}\left(B_1\lambda_1 \frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) \overset{?}{\leqslant} 2A_1 \frac{\beta}{\beta^2 - 1} + 2\,\mathrm{Re}(B_1\lambda_1^{2n-1}).$$

For the left side, we can write for $n \geqslant 1$

$$\begin{aligned} 8 + 4\,\mathrm{Re}\left(B_1\lambda_1 \frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) &\leqslant 8 + 4|B_1||\lambda_1| \frac{|\lambda_1|^{2n-2} + 1}{|\lambda_1^2 - 1|} \\ &\leqslant 8 + 4|B_1||\lambda_1| \frac{2}{|\lambda_1^2 - 1|}. \end{aligned}$$

For the right side, we can write for $n \geqslant 1$

$$\begin{aligned} 2A_1 \frac{\beta}{\beta^2 - 1} + 2\,\mathrm{Re}(B_1\lambda_1^{2n-1}) &\geqslant 2A_1 \frac{\beta}{\beta^2 - 1} - 2|B_1||\lambda_1|^{2n-1} \\ &\geqslant 2A_1 \frac{\beta}{\beta^2 - 1} - 2|B_1||\lambda_1|. \end{aligned}$$

Since the inequality

$$8 + 4|B_1||\lambda_1| \frac{2}{|\lambda_1^2 - 1|} \quad \leqslant \quad 2A_1 \frac{\beta}{\beta^2 - 1} - 2|B_1||\lambda_1|$$

holds true for the given values, we obtain $A \leqslant \frac{3}{2}$.

Next, we will show that $B \leqslant \frac{3}{2}$.

For all $n \geqslant 0$ we have to show that

$$\begin{aligned}
\frac{1 + A_1 \sum_{k=0}^{n} \beta^{2k} + B_1 \sum_{k=0}^{n} \lambda_1^{2k} + C_1 \sum_{k=0}^{n} \lambda_2^{2k}}{A_1 \beta^{2n} + B_1 \lambda_1^{2n} + C_1 \lambda_2^{2n}} \quad &\leqslant^? \quad \frac{3}{2}, \\
2 + 2A_1 \sum_{k=0}^{n-1} \beta^{2k} + 4 \operatorname{Re} \left(B_1 \sum_{k=0}^{n-1} \lambda_1^{2k} \right) \quad &\leqslant^? \quad A_1 \beta^{2n} + 2 \operatorname{Re} \left(B_1 \lambda_1^{2n} \right), \\
2 + 2A_1 \frac{\beta^{2n} - 1}{\beta^2 - 1} + 4 \operatorname{Re} \left(B_1 \frac{\lambda_1^{2n} - 1}{\lambda_1^2 - 1} \right) \quad &\leqslant^? \quad A_1 \beta^{2n} + 2 \operatorname{Re} \left(B_1 \lambda_1^{2n} \right).
\end{aligned}$$

Since

$$\frac{2}{\beta^2 - 1} \leqslant 1,$$

we need to prove the inequality in the form

$$2 + 4 \operatorname{Re} \left(B_1 \frac{\lambda_1^{2n} - 1}{\lambda_1^2 - 1} \right) \quad \leqslant^? \quad A_1 \frac{2}{\beta^2 - 1} + 2 \operatorname{Re} \left(B_1 \lambda_1^{2n} \right).$$

For the left side, we can write for $n \geqslant 0$

$$\begin{aligned}
2 + 4 \operatorname{Re} \left(B_1 \frac{\lambda_1^{2n} - 1}{\lambda_1^2 - 1} \right) \quad &\leqslant \quad 2 + 4|B_1| \frac{|\lambda_1|^{2n} + 1}{|\lambda_1^2 - 1|} \\
&\leqslant \quad 2 + 4|B_1| \frac{2}{|\lambda_1^2 - 1|}.
\end{aligned}$$

For the right side, we can write for $n \geqslant 0$

$$\begin{aligned}
A_1 \frac{2}{\beta^2 - 1} + 2 \operatorname{Re} \left(B_1 \lambda_1^{2n} \right) \quad &\geqslant \quad A_1 \frac{2}{\beta^2 - 1} - 2|B_1||\lambda_1|^{2n} \\
&\geqslant \quad A_1 \frac{2}{\beta^2 - 1} - 2|B_1|.
\end{aligned}$$

Since the inequality

$$2 + 4|B_1| \frac{2}{|\lambda_1^2 - 1|} \quad \leqslant \quad A_1 \frac{2}{\beta^2 - 1} - 2|B_1|$$

holds true for given values, we obtain $B \leqslant \frac{3}{2}$.

3. Computation of C and D. The sequence $d_n := |\nu(\varphi^n(01))|$ satisfies $d_0 = 4, d_1 = 7, d_2 = 13$, and the recurrence relation $d_n = 2d_{n-1} - d_{n-2} + d_{n-3}$.

   The explicit solution reads

$$d_n = A_2\beta^n + B_2\lambda_1^n + C_2\lambda_2^n,$$

   where

$$\beta \doteq 1.75488, \quad \lambda_1 \doteq 0.12256 + 0.74486i, \quad \lambda_2 = \overline{\lambda_1}$$

   are the roots of the polynomial $t^3 - 2t^2 + t - 1$, and

$$\begin{aligned} A_2 &= \frac{4|\lambda_1|^2 - 14\,\mathrm{Re}(\lambda_1) + 13}{|\beta - \lambda_1|^2} \doteq 4.213205567\,; \\ B_2 &= \frac{4\beta\lambda_2 - 7(\beta + \lambda_2) + 13}{(\beta - \lambda_1)(\lambda_2 - \lambda_1)} \doteq -0.106602784 + 0.24671731i\,; \\ C_2 &= \overline{B_2}\,. \end{aligned}$$

   First, we will show that $C = \frac{3}{2}$. Recall that

$$C = \sup\left\{\frac{|01\nu(01\varphi^2(01)\varphi^4(01)\ldots\varphi^{2n}(01))|}{|\nu(\varphi^{2n}(01))|} : n \geqslant 1\right\} \cup \left\{\frac{|1\nu(10)0|}{|\nu(10)|}\right\},$$

   consequently, $C \geqslant \frac{|1\nu(10)0|}{|\nu(10)|} = \frac{3}{2}$.
   It suffices to show for all $n \geqslant 1$ that

$$\begin{aligned} \frac{2 + A_2\sum_{k=0}^{n}\beta^{2k} + B_2\sum_{k=0}^{n}\lambda_1^{2k} + C_2\sum_{k=0}^{n}\lambda_2^{2k}}{A_2\beta^{2n} + B_2\lambda_1^{2n} + C_2\lambda_2^{2n}} &\leqslant^? \quad \frac{3}{2}, \\ 4 + 2A_2\sum_{k=0}^{n-1}\beta^{2k} + 4\,\mathrm{Re}\left(B_2\sum_{k=0}^{n-1}\lambda_1^{2k}\right) &\leqslant^? \quad A_2\beta^{2n} + 2\,\mathrm{Re}\left(B_2\lambda_1^{2n}\right), \\ 4 + 2A_2\frac{\beta^{2n}-1}{\beta^2-1} + 4\,\mathrm{Re}\left(B_2\frac{\lambda_1^{2n}-1}{\lambda_1^2-1}\right) &\leqslant^? \quad A_2\beta^{2n} + 2\,\mathrm{Re}\left(B_2\lambda_1^{2n}\right). \end{aligned}$$

   Since

$$\frac{2}{\beta^2 - 1} \leqslant 1,$$

   we need to prove the inequality in the form

$$4 + 4\,\mathrm{Re}\left(B_2\frac{\lambda_1^{2n}-1}{\lambda_1^2-1}\right) \quad \leqslant^? \quad A_2\frac{2}{\beta^2-1} + 2\,\mathrm{Re}\left(B_2\lambda_1^{2n}\right).$$

Now, we need to be more careful with the approximations. For the left side, we can write for $n \geqslant 2$

$$\begin{aligned} 4 + 4\operatorname{Re}\left(B_2 \frac{\lambda_1^{2n} - 1}{\lambda_1^2 - 1}\right) &\leqslant 4 + 4\operatorname{Re}\left(\frac{B_2}{1 - \lambda_1^2}\right) + 4|B_2| \frac{|\lambda_1|^{2n}}{|1 - \lambda_1^2|} \\ &\leqslant 4 + 4\operatorname{Re}\left(\frac{B_2}{1 - \lambda_1^2}\right) + 4|B_2| \frac{|\lambda_1|^{4}}{|1 - \lambda_1^2|}. \end{aligned}$$

For the right side, we can write for $n \geqslant 2$

$$\begin{aligned} A_2 \frac{2}{\beta^2 - 1} + 2\operatorname{Re}\left(B_2 \lambda_1^{2n}\right) &\geqslant A_2 \frac{2}{\beta^2 - 1} - 2|B_2||\lambda_1|^{2n} \\ &\geqslant A_2 \frac{2}{\beta^2 - 1} - 2|B_2||\lambda_1|^{4}. \end{aligned}$$

Since the inequality

$$4 + 4\operatorname{Re}\left(\frac{B_2}{1 - \lambda_1^2}\right) + 4|B_2| \frac{|\lambda_1|^{4}}{|1 - \lambda_1^2|} \leqslant A_2 \frac{2}{\beta^2 - 1} - 2|B_2||\lambda_1|^{4}$$

holds true for given values, it remains to check the case for $n = 1$.

If $n = 1$, we get

$$\frac{2 + d_0 + d_2}{d_2} = \frac{19}{13} < \frac{3}{2}.$$

Therefore, we have proven that $C = \frac{3}{2}$.

It remains to prove $D \leqslant \frac{3}{2}$, however, the steps are the same as in the proof of the inequality $A \leqslant \frac{3}{2}$. Thus, we dare to skip it.

We have shown that $\max\{A, B, C, D\} = \frac{3}{2}$, and $F = 1$. Consequently, $E(\nu(\mathbf{p})) = 1 + \max\{A, B, C, D, F\} = \frac{5}{2}$.

□

4.3 The infinite word $\boldsymbol{\mu}(\mathbf{p})$

The morphism μ has the form:

$$\begin{aligned} \mu(0) &= \mathtt{011001}, \\ \mu(1) &= \mathtt{1001}, \\ \mu(2) &= \mathtt{0}. \end{aligned}$$

$$\mu(\mathbf{p}) = \mathtt{0110011001010010110010100101100110010110010100101100110010100101100110010110010}\ldots$$

and μ is injective.

Remark 27. The reader may easily check that any factor of $\mu(\mathbf{p})$ of length at least six has a synchronization point.

Using the above remark and Theorem 9, we deduce that

$$E^*(\mu(\mathbf{p})) = E^*(\mathbf{p}).$$

4.3.1 Bispecial factors in $\boldsymbol{\mu}(\mathbf{p})$

Lemma 28. *Let $v \in \mathcal{L}(\mu(\mathbf{p}))$ be a BS factor of length at least six. Then one of the items holds.*

1. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \mu(w)\mathtt{01}$ and $\mathtt{0}w, \mathtt{2}w, w\mathtt{0}, w\mathtt{2} \in \mathcal{L}(\mathbf{p})$.*
2. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \mathtt{011001}\mu(w)$ and $\mathtt{0}w, \mathtt{1}w, w\mathtt{1}, w\mathtt{2} \in \mathcal{L}(\mathbf{p})$.*
3. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \mu(w)$ and $\mathtt{0}w, \mathtt{2}w, w\mathtt{1}, w\mathtt{2} \in \mathcal{L}(\mathbf{p})$.*
4. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \mathtt{011001}\mu(w)\mathtt{01}$ and $\mathtt{0}w, \mathtt{1}w, w\mathtt{0}, w\mathtt{2} \in \mathcal{L}(\mathbf{p})$.*

We would like to point out that in this section, we use the same notation for BS factors and their shortest return words as in Section 4.2. We are persuaded that no confusion arises since we do not refer here to the BS factors and their shortest return words from Section 4.2.

Corollary 29. *Let v be a BS factor in $\mu(\mathbf{p})$ of length at least six. Then $v = \mathtt{011001}$ or $v = \mathtt{100101}$ or $v = \mathtt{01100101}$ or v has one of the following forms:*

($\mathcal{A}$)

$$v_A^{(n)} = \mu(\mathtt{1}\varphi^2(\mathtt{1})\varphi^4(\mathtt{1})\cdots\varphi^{2n}(\mathtt{1})\varphi^{2n-1}(\mathtt{0})\varphi^{2n-3}(\mathtt{0})\cdots\varphi(\mathtt{0}))\mathtt{01}$$

for $n \geqslant 1$.

$v_A^{(n)}$ and $\mathtt{01}\mu(\mathtt{1}\varphi(\mathtt{012})\varphi^3(\mathtt{012})\ldots\varphi^{2n-1}(\mathtt{012}))$ have the same Parikh vector.

($\mathcal{B}$)

$$\begin{aligned} v_B^{(n)} &= \mathtt{011001}\mu(\varphi(\mathtt{1})\varphi^3(\mathtt{1})\cdots\varphi^{2n+1}(\mathtt{1})\varphi^{2n}(\mathtt{0})\varphi^{2n-2}(\mathtt{0})\cdots\varphi^2(\mathtt{0})\mathtt{0}) \\ &= \mu(\mathtt{0}\varphi(\mathtt{1})\varphi^3(\mathtt{1})\cdots\varphi^{2n+1}(\mathtt{1})\varphi^{2n}(\mathtt{0})\varphi^{2n-2}(\mathtt{0})\cdots\varphi^2(\mathtt{0})\mathtt{0}) \end{aligned}$$

for $n \geqslant 0$.

$v_B^{(n)}$ and $\mathtt{000111}\mu(\mathtt{012}\varphi^2(\mathtt{012})\varphi^4(\mathtt{012})\ldots\varphi^{2n}(\mathtt{012}))$ have the same Parikh vector.

($\mathcal{C}$)

$$v_C^{(n)} = \mu(\mathtt{1}\varphi^2(\mathtt{1})\varphi^4(\mathtt{1})\cdots\varphi^{2n}(\mathtt{1})\varphi^{2n}(\mathtt{0})\varphi^{2n-2}(\mathtt{0})\cdots\varphi^2(\mathtt{0})\mathtt{0})$$

for $n \geqslant 0$.

$v_C^{(n)}$ and $\mu(\mathtt{01}\varphi^2(\mathtt{01})\varphi^4(\mathtt{01})\ldots\varphi^{2n}(\mathtt{01}))$ have the same Parikh vector.

($\mathcal{D}$)

$$v_D^{(n)} = \mathtt{011001}\mu(\varphi(\mathtt{1})\varphi^3(\mathtt{1})\cdots\varphi^{2n+1}(\mathtt{1})\varphi^{2n+1}(\mathtt{0})\varphi^{2n-1}(\mathtt{0})\cdots\varphi(\mathtt{0}))\mathtt{01}$$

for $n \geqslant 0$.

$v_D^{(n)}$ and $\mathtt{00001111}\mu(\varphi(\mathtt{01})\varphi^3(\mathtt{01})\ldots\varphi^{2n+1}(\mathtt{01}))$ have the same Parikh vector.

4.3.2 The shortest return words to bispecial factors in $\mu(\mathbf{p})$

Lemma 30. *If w is a BS factor of $\mathbf{p}$, $|w| \geqslant 2$, and v is its return word, then $\mu(v)$ is a return word to $\mu(w)$.*

Proof. On one hand, since vw contains w as a prefix and as a suffix, then $\mu(v)\mu(w)$ contains $\mu(w)$ as a prefix and as a suffix, too. On the other hand, w starts in $\mathtt{10}$ or $\mathtt{21}$ and ends in $\mathtt{0}$ or $\mathtt{1}$, therefore $\mu(w)$ has the following synchronization points $\bullet\mu(w)\bullet$. Consequently, $\mu(v)\mu(w)$ cannot contain $\mu(w)$ somewhere in the middle because in such a case, by injectivity of μ, vw would contain w also somewhere in the middle. □

Applying Lemma 30 and Observation 18, we have the following description of the shortest return words to BS factors.

Corollary 31. *The shortest return words to BS factors of length greater than eight in $\mu(\mathbf{p})$ have the following properties.*

($\mathcal{A}$) *The shortest return word $\hat{r}_A^{(n)}$ to $v_A^{(n)}$ has the same Parikh vector as $\mu(\varphi^{2n-1}(\mathtt{012}))$ for $n \geqslant 1$.*

($\mathcal{B}$) *The shortest return word $\hat{r}_B^{(n)}$ to $v_B^{(n)}$ has the same Parikh vector as $\mu(\varphi^{2n}(\mathtt{012}))$.*

($\mathcal{C}$) *The shortest return word $\hat{r}_C^{(n)}$ to $v_C^{(n)}$ has the same Parikh vector as $\mu(\varphi^{2n}(\mathtt{01}))$.*

($\mathcal{D}$) *The shortest return word $\hat{r}_D^{(n)}$ to $v_D^{(n)}$ has the same Parikh vector as $\mu(\varphi^{2n+1}(\mathtt{01}))$.*

Proof. We will prove case ($\mathcal{A}$). The other cases are similar. Let us consider $n \geqslant 1$ and the bispecial factor

$$v_A^{(n)} = \mu(\mathtt{1}\varphi^2(\mathtt{1})\varphi^4(\mathtt{1})\cdots\varphi^{2n}(\mathtt{1})\varphi^{2n-1}(\mathtt{0})\varphi^{2n-3}(\mathtt{0})\cdots\varphi(\mathtt{0}))\mathtt{01} = \mu(w_A^{(n)})\mathtt{01}.$$

Using Corollary 20, we know that the shortest return word to $w_A^{(n)}$ has the same Parikh vector as $\varphi^{2n-1}(\mathtt{012})$, moreover the shortest return word is a prefix of all of the return words.

Using Lemma 30, and the fact that μ is non-erasing, we obtain that the shortest return word to $\mu(w_A^{(n)})$ has the same Parikh vector as $\mu(\varphi^{2n-1}(\mathtt{012}))$. Using Observation 18 twice, we obtain that the shortest return word to $\mu(w_A^{(n)})\mathtt{01}$ has the same Parikh vector as $\mu(\varphi^{2n-1}(\mathtt{012}))$. □

4.3.3 The critical exponent of $\mu(\mathbf{p})$

Using Theorem 8 and the description of BS factors from Corollary 29 and of their shortest return words from Corollary 31, we obtain the following formula for the critical exponent of $\mu(\mathbf{p})$.

$$E(\mu(\mathbf{p})) = 1 + \max\{A, B, C, D, F\}\ ,$$

where

$$A = \sup\left\{\frac{|v_A^{(n)}|}{|\hat{r}_A^{(n)}|} : n \geqslant 1\right\} = \sup\left\{\frac{|\mathtt{01}\mu(\mathtt{1}\varphi(\mathtt{012})\varphi^3(\mathtt{012})\cdots\varphi^{2n-1}(\mathtt{012}))|}{|\mu(\varphi^{2n-1}(\mathtt{012}))|} : n \geqslant 1\right\};$$

$$B = \sup\left\{\frac{|v_B^{(n)}|}{|\hat{r}_B^{(n)}|} : n \geqslant 0\right\} = \sup\left\{\frac{|\mathtt{000111}\mu(\mathtt{012}\varphi^2(\mathtt{012})\varphi^4(\mathtt{012})\cdots\varphi^{2n}(\mathtt{012}))|}{|\mu(\varphi^{2n}(\mathtt{012}))|} : n \geqslant 0\right\};$$

$$C = \sup\left\{\frac{|v_C^{(n)}|}{|\hat{r}_C^{(n)}|} : n \geqslant 0\right\} = \sup\left\{\frac{|\mu(\mathtt{01}\varphi^2(\mathtt{01})\varphi^4(\mathtt{01})\cdots\varphi^{2n}(\mathtt{01}))|}{|\mu(\varphi^{2n}(\mathtt{01}))|} : n \geqslant 1\right\};$$

$$D = \sup\left\{\frac{|v_D^{(n)}|}{|\hat{r}_D^{(n)}|} : n \geqslant 0\right\} = \sup\left\{\frac{|\mathtt{00001111}\mu(\varphi(\mathtt{01})\varphi^3(\mathtt{01})\cdots\varphi^{2n+1}(\mathtt{01}))|}{|\mu(\varphi^{2n+1}(\mathtt{01}))|} : n \geqslant 0\right\};$$

$$F = \max\left\{\frac{|w|}{|r|} : w \text{ BS in } \mu(\mathbf{p}) \text{ of length at most 8 and } r \text{ its shortest return word}\right\}.$$

Theorem 32. *The critical exponent of $\mu(\mathbf{p})$ equals*

$$E(\mu(\mathbf{p})) = \frac{28}{11}.$$

Proof. To evaluate the critical exponent of $\mu(\mathbf{p})$ using the above formula, we have to do several steps.

1. Determining the shortest return words of BS factors of length at most 8 in $\mu(\mathbf{p})$:
   - `0` is a BS factor with the shortest return word `0`.
   - `1` is a BS factor with the shortest return word `1`.
   - `01` is a BS factor with the shortest return word `01`.
   - `10` is a BS factor with the shortest return word `10`.
   - `010` is a BS factor with the shortest return word `01`.
   - `1001` is a BS factor with the shortest return word `1001`.
   - `011001` is a BS factor with the shortest return word `0110`.
   - `100101` is a BS factor with the shortest return word `10010`.
   - `01100101` is a BS factor with the shortest return word `011001`.

   Therefore, $F = \max\left\{1, \frac{3}{2}, \frac{6}{5}, \frac{8}{6}\right\} < \frac{17}{11}$.

2. Computation of A and B. The sequence $e_n := |\mu(\varphi^n(\mathtt{012}))|$ satisfies $e_0 = 11, e_1 = 21, e_2 = 36$, and the recurrence relation $e_n = 2e_{n-1} - e_{n-2} + e_{n-3}$.

   The explicit solution reads

$$e_n = A_3\beta^n + B_3\lambda_1^n + C_3\lambda_2^n,$$

where

$$\beta \doteq 1.75488, \quad \lambda_1 \doteq 0.12256 + 0.74486i, \quad \lambda_2 = \overline{\lambda_1}$$

are the roots of the polynomial $t^3 - 2t^2 + t - 1$, and

$$\begin{aligned} A_3 &= \frac{11|\lambda_1|^2 - 42\,\mathrm{Re}(\lambda_1) + 36}{|\beta - \lambda_1|^2} \doteq 11.530751580\,; \\ B_3 &= \frac{11\beta\lambda_2 - 21(\beta + \lambda_2) + 36}{(\beta - \lambda_1)(\lambda_2 - \lambda_1)} \doteq -0.265375790 - 0.557144391i\,; \\ C_3 &= \overline{B_3}\,. \end{aligned}$$

Let us show that $A \leqslant \frac{17}{11}$. We have to show for all $n \geqslant 1$ that

$$\begin{aligned} \frac{6 + A_3 \sum_{k=1}^{n} \beta^{2k-1} + B_3 \sum_{k=1}^{n} \lambda_1^{2k-1} + C_3 \sum_{k=1}^{n} \lambda_2^{2k-1}}{A_3\beta^{2n-1} + B_3\lambda_1^{2n-1} + C_3\lambda_2^{2n-1}} &\overset{?}{\leqslant} \frac{17}{11}, \\ 66 + 11A_3 \sum_{k=1}^{n} \beta^{2k-1} + 22\,\mathrm{Re}\left(B_3 \sum_{k=1}^{n} \lambda_1^{2k-1}\right) &\overset{?}{\leqslant} 17A_3\beta^{2n-1} + 34\,\mathrm{Re}(B_2\lambda_1^{2n-1}), \\ 66 + 11A_3 \sum_{k=1}^{n-1} \beta^{2k-1} + 22\,\mathrm{Re}\left(B_3 \sum_{k=1}^{n-1} \lambda_1^{2k-1}\right) &\overset{?}{\leqslant} 6A_3\beta^{2n-1} + 12\,\mathrm{Re}(B_3\lambda_1^{2n-1}), \\ 66 + 11A_3\left(\frac{\beta^{2n-1}}{\beta^2 - 1} - \frac{\beta}{\beta^2 - 1}\right) + 22\,\mathrm{Re}\left(B_3\lambda_1 \frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) &\overset{?}{\leqslant} 6A_3\beta^{2n-1} + 12\,\mathrm{Re}(B_3\lambda_1^{2n-1}). \end{aligned}$$

Since

$$\frac{11}{\beta^2 - 1} \leqslant 6,$$

we need to prove the inequality in the form

$$66 + 22\,\mathrm{Re}\left(B_3\lambda_1 \frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) \overset{?}{\leqslant} 11A_3 \frac{\beta}{\beta^2 - 1} + 12\,\mathrm{Re}(B_3\lambda_1^{2n-1}).$$

For the left side, we can write for $n \geqslant 1$

$$\begin{aligned} 66 + 22\,\mathrm{Re}\left(B_3\lambda_1 \frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) &\leqslant 66 + 22|B_3||\lambda_1| \frac{|\lambda_1|^{2n-2} + 1}{|\lambda_1^2 - 1|} \\ &\leqslant 66 + 22|B_3||\lambda_1| \frac{2}{|\lambda_1^2 - 1|}. \end{aligned}$$

For the right side, we can write for $n \geqslant 1$

$$11A_3 \frac{\beta}{\beta^2 - 1} + 12\,\mathrm{Re}(B_3\lambda_1^{2n-1}) \geqslant 11A_3 \frac{\beta}{\beta^2 - 1} - 12|B_3||\lambda_1|^{2n-1}$$

$$\geqslant \quad 11A_3\frac{\beta}{\beta^2-1} - 12|B_3||\lambda_1|.$$

Since the inequality

$$66 + 22|B_3||\lambda_1|\frac{2}{|\lambda_1^2-1|} \quad \leqslant \quad 11A_3\frac{\beta}{\beta^2-1} - 12|B_1||\lambda_1|$$

holds true for the given values, we obtain $A \leqslant \frac{17}{11}$.
Next, we will show that $B \leqslant \frac{17}{11}$.
Since for $n = 0$ we have $\frac{|v_B^{(0)}|}{|\hat{r}_B^{(0)}|} = \frac{6+11}{11} = \frac{17}{11}$, it remains to show that for all $n \geqslant 1$

$$\frac{6 + A_3\sum_{k=0}^{n}\beta^{2k} + B_3\sum_{k=0}^{n}\lambda_1^{2k} + C_3\sum_{k=0}^{n}\lambda_2^{2k}}{A_3\beta^{2n} + B_3\lambda_1^{2n} + C_3\lambda_2^{2n}} \quad \leqslant^? \quad \frac{17}{11},$$

$$66 + 11A_3\sum_{k=0}^{n-1}\beta^{2k} + 22\operatorname{Re}\left(B_3\sum_{k=0}^{n-1}\lambda_1^{2k}\right) \quad \leqslant^? \quad 6A_3\beta^{2n} + 12\operatorname{Re}\left(B_3\lambda_1^{2n}\right),$$

$$66 + 11A_3\frac{\beta^{2n}-1}{\beta^2-1} + 22\operatorname{Re}\left(B_3\frac{\lambda_1^{2n}-1}{\lambda_1^2-1}\right) \quad \leqslant^? \quad 6A_3\beta^{2n} + 12\operatorname{Re}\left(B_3\lambda_1^{2n}\right).$$

Now, we need to be more careful with the approximations, we will therefore prove the inequality in the form

$$66 + 22\operatorname{Re}\left(B_3\frac{\lambda_1^{2n}-1}{\lambda_1^2-1}\right) \quad \leqslant^? \quad \frac{11A_3}{\beta^2-1} + A_3\beta^{2n}\left(6 - \frac{11}{\beta^2-1}\right) + 12\operatorname{Re}\left(B_3\lambda_1^{2n}\right).$$

For the left side, we can write for $n \geqslant 1$

$$\begin{aligned} 66 + 22\operatorname{Re}\left(B_3\frac{\lambda_1^{2n}-1}{\lambda_1^2-1}\right) \quad &\leqslant \quad 66 + 22|B_3|\frac{|\lambda_1|^{2n}+1}{|\lambda_1^2-1|} \\ &\leqslant \quad 66 + 22|B_3|\frac{1+|\lambda_1|^2}{|\lambda_1^2-1|}. \end{aligned}$$

For the right side, we can write for $n \geqslant 1$

$$\begin{aligned} \frac{11A_3}{\beta^2-1} + A_3\beta^{2n}\left(6 - \frac{11}{\beta^2-1}\right) + 12\operatorname{Re}\left(B_3\lambda_1^{2n}\right) \; &\geqslant \; \frac{11A_3}{\beta^2-1} + A_3\beta^2\left(6 - \frac{11}{\beta^2-1}\right) - 12|B_3||\lambda_1|^{2n} \\ &\geqslant \; \frac{11A_3}{\beta^2-1} + A_3\beta^2\left(6 - \frac{11}{\beta^2-1}\right) - 12|B_3||\lambda_1|^2. \end{aligned}$$

Since the inequality

$$66 + 22|B_3|\frac{1+|\lambda_1|^2}{|\lambda_1^2-1|} \quad \leqslant \quad \frac{11A_3}{\beta^2-1} + A_3\beta^2\left(6 - \frac{11}{\beta^2-1}\right) - 12|B_3||\lambda_1|^2$$

holds true for the given values, we conclude $B = \frac{17}{11}$.

3. Computation of C and D. The sequence $f_n := |\mu(\varphi^n(01))|$ satisfies $f_0 = 10, f_1 = 15, f_2 = 26$, and the recurrence relation $f_n = 2f_{n-1} - f_{n-2} + f_{n-3}$.

   The explicit solution reads

$$f_n = A_4\beta^n + B_4\lambda_1^n + C_4\lambda_2^n,$$

   where

$$\beta \doteq 1.75488, \quad \lambda_1 \doteq 0.12256 + 0.74486i, \quad \lambda_2 = \overline{\lambda_1}$$

   are the roots of the polynomial $t^3 - 2t^2 + t - 1$, and

$$\begin{aligned} A_4 &= \frac{10|\lambda_1|^2 - 30\,\mathrm{Re}(\lambda_1) + 26}{|\beta - \lambda_1|^2} \doteq 8.704306843\,; \\ B_4 &= \frac{10\beta\lambda_2 - 15(\beta + \lambda_2) + 26}{(\beta - \lambda_1)(\lambda_2 - \lambda_1)} \doteq 0.647846579 + 0.291191845i\,; \\ C_4 &= \overline{B_4}\,. \end{aligned}$$

   The computation for $C \leqslant \frac{17}{11}$ is the same as for B. Let us show that $D \leqslant \frac{17}{11}$. We have to show for $n \geqslant 1$ that

$$\begin{aligned} \frac{8 + A_4\sum_{k=1}^n \beta^{2k-1} + B_4\sum_{k=1}^n \lambda_1^{2k-1} + C_4\sum_{k=1}^n \lambda_2^{2k-1}}{A_4\beta^{2n-1} + B_4\lambda_1^{2n-1} + C_4\lambda_2^{2n-1}} &\leqslant^? \frac{17}{11}, \\ 88 + 11A_4\left(\frac{\beta^{2n-1}}{\beta^2 - 1} - \frac{\beta}{\beta^2 - 1}\right) + 22\,\mathrm{Re}\left(B_4\lambda_1\frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) &\leqslant^? 6A_4\beta^{2n-1} + 12\,\mathrm{Re}(B_4\lambda_1^{2n-1}). \end{aligned}$$

   Since

$$\frac{11}{\beta^2 - 1} \leqslant 6,$$

   we need to prove the inequality in the form

$$88 + 22\,\mathrm{Re}\left(B_4\lambda_1\frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) \leqslant^? 11A_4\frac{\beta}{\beta^2 - 1} + 12\,\mathrm{Re}(B_4\lambda_1^{2n-1}).$$

   For the left side, we can write for $n \geqslant 1$

$$\begin{aligned} 88 + 22\,\mathrm{Re}\left(B_4\lambda_1\frac{1 - \lambda_1^{2n-2}}{1 - \lambda_1^2}\right) &\leqslant 88 + 22\,\mathrm{Re}\left(\frac{B_4\lambda_1}{1 - \lambda_1^2}\right) + 22|B_4|\frac{|\lambda_1|^{2n-1}}{|1 - \lambda_1^2|} \\ &\leqslant 88 + 22\,\mathrm{Re}\left(\frac{B_4\lambda_1}{1 - \lambda_1^2}\right) + 22|B_4|\frac{|\lambda_1|}{|1 - \lambda_1^2|}. \end{aligned}$$

   For the right side, we can write for $n \geqslant 1$

$$11A_4\frac{\beta}{\beta^2 - 1} + 12\,\mathrm{Re}(B_4\lambda_1^{2n-1}) \geqslant 11A_4\frac{\beta}{\beta^2 - 1} - 12|B_4||\lambda_1|^{2n-1}$$

$$\geqslant \quad 11A_4\frac{\beta}{\beta^2-1} - 12|B_4||\lambda_1|.$$

Since the inequality

$$88 + 22\,\mathrm{Re}\left(\frac{B_4\lambda_1}{1-\lambda_1^2}\right) + 22|B_4|\frac{|\lambda_1|}{|1-\lambda_1^2|} \quad \leqslant \quad 11A_4\frac{\beta}{\beta^2-1} - 12|B_4||\lambda_1|$$

holds true for the given values, we obtain $D \leqslant \frac{17}{11}$.

We have shown that $\max\{A,B,C,D\} = B = \frac{17}{11}$, and $F < \frac{17}{11}$. Consequently, $E(\mu(\mathbf{p})) = 1 + \max\{A,B,C,D,F\} = \frac{28}{11}$. □

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