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Abstract. Temporal graphs are graphs that evolve over time. Many problems which are polynomial-time solvable in standard graphs become NP-hard when appropriately defined in the realm of temporal graphs. This suggested the definition of several parameters for temporal graphs and to prove the fixed-parameter tractability of several problems with respect to these parameters. In this paper, we introduce a hierarchy of parameters based on the previously defined interval membership width and on the temporal evolution of the connected components of the underlying static graph. We then show that the Eulerian trail problem and the temporal 2-coloring problem are both fixed-parameter tractable (in short, FPT) with respect to any of the parameters in the hierarchy. We also introduce a vertex-variant of the parameters and we show that the firefighter problem (which was known to be FPT with respect to the vertex-variant of the interval membership width) is also FPT with respect to one of the parameters in the second level of the hierarchy.

Keywords: Temporal graphs, Eulerian trails, Temporal coloring, Firefighter, Parameterized complexity, Width measures

1 Introduction

A temporal graph is a graph whose underlying topology is subject to discrete changes over time. Several real-world networks, such as social networks, transportation networks, and information and communication networks, can be modeled as temporal graphs. This is usually done by associating time labels to the edges of a graph, in order to indicate the moments of existence of the edges themselves (with the vertex set of the graph remaining unchanged).

A temporal graph $\mathcal{G}$ with lifetime τ is a pair $(G = (V, E), \lambda)$, where $\lambda : E \to 2^{[\tau]}$ is a *time-labeling* function that assigns a set of integer time labels to each edge of the graph G.[5] An edge $e \in E$ is available only at the times specified by $\lambda(e)$. Due to their relevance and applicability in many areas, temporal graphs have attracted a lot of attention in the past decade (we refer the reader to the book of Holme and Saramäki [13], to the survey of Michail [18], and to the seminal paper of Kempe, Kleinberg, and Kumar [14]). Temporal graphs have also appeared under different names in the literature, such as time-varying graphs [10], dynamic graphs [5], evolving networks [2], and link streams [15].

Paths and walks in a temporal graph have to traverse a sequence of adjacent edges $e_1, \ldots, e_k$ at increasing times $t_1 < \ldots < t_k$, respectively, with $t_i \in \lambda(e_i)$ for every $i \in [k]$.[6] By referring to this kind of paths and walks, several polynomial-time problems on standard graphs become intractable when transferred to the temporal realm, such as, for example, the computation of connected components [1] and the identification of Eulerian walks [17]. Moreover, the introduction of the time dimension suggests new problems connected to paths and walks, such as, for example, determining the existence of a restless (that is, respecting specific waiting constraints) connection between two nodes, which is intractable for paths and polynomial-time solvable for walks [6], and the analysis of simple spreading processes [9].

The temporal version of other classical graph problems have been considered in the last ten years, such as, for example, the well-known coloring problem. The temporal coloring problem consists in deciding whether there exists a coloring of the temporal nodes (that is, the nodes at different time steps) such that each edge of the graph is properly colored in at least one time step. It is known that deciding whether a temporal graph admits a temporal coloring using 2 colors is NP-complete (even under very strict constraints of the temporal graph) [16].

When dealing with temporal graphs, the intractability of several problems holds also when the underlying static graph has small well-known parameters, such as the tree-width or the feedback vertex/edge number. This prompts the need to develop parameters that not only consider the underlying graph structure but also account the temporal structure of the input graph. Some such measures, like temporal variations of the above two parameters, have already been proposed [6,11]. In this paper, we focus on the parameter introduced by Bumpus and Meeks [3], called *interval membership width*, which intuitively quantifies the extent to which the set of intervals defined by the first and last appearance of each edge can overlap. The value of this parameter, hence, does not depend on the structure of the underlying static graph (other than its number of edges), but it is instead influenced only by the temporal structure of the input graph. In [3] it is shown that Temporal Eulerian Trail is FPT when parameterized by the interval membership width, while in [12] it is shown that

[5] For every positive integer k, $[k]$ denotes the set $\{1, 2, \ldots, k\}$.

[6] Similarly, one can consider non-decreasing sequences, i.e. with $t_1 \leq \ldots \leq t_k$. In this paper, however, we focus on the strictly increasing case.

Temporal Firefighter Reserve is FPT when parameterized by its 'vertex variant' (see below for the definitions of all considered problems).

Motivated by these latter results, we here modify the definition of the interval membership width by taking into account the evolution (both forward and backward) of the connected components of the temporal graph and by introducing a new parameter called *connected interval membership width* (together with its vertex-variant). We show that both Temporal Eulerian Trail and Temporal 2-Coloring are FPT with respect to this new parameter. We also show that Temporal Firefighter Reserve is FPT with respect to its vertex-variant. We then introduce another parameter which is based on the search for the best combination of the forward and the backward connected interval membership width and we prove that Temporal Eulerian Trail and Temporal 2-Coloring are FPT with respect to this latter parameter.

Preliminaries We use standard definitions and notation of graph theory (we refer the unfamiliar reader to [20]). We will also make use of the following notations. Given an undirected graph $G = (V, E)$ and a vertex $v \in V$, the *neighborhood* of v is defined as $N_G(v) = \{u \mid \{u, v\} \in E\}$. For a set $X \subseteq V(G)$, we also use $N_G(X)$ to denote the set $\bigcup_{x \in X} N_G(x) \setminus X$. We omit G when it is clear from the context. For any edge set $A \subseteq E$, $V(A)$ denotes the set of vertices with an incident edge in A, that is, $V(A) = \bigcup_{e=\{u,v\} \in A}\{u, v\}$. For any connected component C of G, $V(C)$ (respectively, $E(C)$) denotes the set of nodes (respectively, edges) in C.

Concerning temporal graphs, we use and extend the notation in [18]. A *temporal graph* $\mathcal{G}$ is a pair (G, λ), where $G = (V, E)$ is the *underlying (undirected) graph* of $\mathcal{G}$, and $\lambda : E \to 2^{\mathbb{N}}$ is a *time-labeling* function which assigns to every edge of G a finite set of integer time labels. Without loss of generality, we can assume that $\min \bigcup_{e \in E} \lambda(e) = 1$. The *lifetime* τ of $\mathcal{G}$ is defined as $\tau = \max \bigcup_{e \in E} \lambda(e)$. For every edge $e = \{u, v\} \in E$ and every $t \in \lambda(e)$, the triple (u, v, t) (or, equivalently, the pair (e, t)) is said to be a *temporal edge* of $\mathcal{G}$.

Given a temporal graph $\mathcal{G} = (G = (V, E), \lambda)$ with lifetime τ, the *snapshot at time* t, with $t \in [\tau]$, is the graph $G_t = (V, E_t)$ where $E_t = \{e \in E : t \in \lambda(e)\}$. The temporal graph $\mathcal{G}$ can then also be defined as the sequence $G_1, \ldots, G_\tau$ of its snapshots. In the following, $\mathcal{G}_{\leq}(t)$ (respectively, $\mathcal{G}_{\geq}(t)$) will denote the temporal graph formed by the first t (respectively, last $\tau - t + 1$) snapshots of $\mathcal{G}$. Moreover, $G_{\leq}(t)$ (respectively, $G_{\geq}(t)$) will denote the underlying graph of $\mathcal{G}_{\leq}(t)$ (respectively, $\mathcal{G}_{\geq}(t)$). For $S \subseteq V$ and $t \in [\tau]$, we define $N_t(S)$ to be the set of all vertices that are temporally adjacent at time t to the vertices in S, excluding S, that is, $N_t(S) = N_{G_t}(S)$. A *(strict) temporal walk* from u to v in $\mathcal{G}$ is a sequence of temporal edges $(u_1, v_1, t_1), \ldots, (u_k, v_k, t_k)$ such that $u_1 = u$, $v_k = v$, and, for any $i \in [k-1]$, $v_i = u_{i+1}$ and $t_i < t_{i+1}$. A temporal walk is said to be a *temporal trail* if no edge in E is traversed twice. A temporal trail is said to be a *temporal path* if no node in V is visited twice.

The following lemma introduces a transformation of a temporal graph that will allow us to easily design a backward version of our algorithms (for the sake of brevity, all proofs have been omitted in this extended abstract).

Lemma 1 ([4]). *Given a temporal graph $\mathcal{G} = (G = (V, E), \lambda)$, let $\mathcal{G}^R = (G = (V, E), \rho)$ be the* reverse temporal graph *of $\mathcal{G}$ obtained by setting, for each $e \in E$, $\rho(e) = \cup_{t \in \lambda(e)}\{-t-1\}$. Then, there exists a temporal walk in $\mathcal{G}$ starting from u and arriving at v at time at most t if and only if there exists a temporal walk in $\mathcal{G}^R$ starting from v at time $-t$ and arriving at u.*

Edge Exploration. An *Eulerian walk* (respectively, *trail*) in a graph is a walk which traverses every edge at least (respectively, exactly) once. A *temporal Eulerian walk* (respectively, *trail*) in a temporal graph is a temporal walk (respectively, trail) $(e_1, t_1), \ldots, (e_m, t_m)$ such that $e_1, \ldots, e_m$ is an Eulerian walk (respectively, trail) in the underlying graph. The Temporal Eulerian Walk (respectively, Temporal Eulerian Trail) problem consists of deciding whether a temporal graph admits a temporal Eulerian walk (respectively, trail). It is known that these two problems are both NP-complete [17]. The following lemma, instead, immediately follows from the proof of Lemma 1.

Lemma 2. *Given a temporal graph $\mathcal{G} = (G = (V, E), \lambda)$ and its reverse temporal graph $\mathcal{G}^R$, there exists an Eulerian walk in $\mathcal{G}$ if and only if there exists an Eulerian walk in $\mathcal{G}^R$.*

Temporal Coloring. Given a temporal graph $\mathcal{G} = (G = (V, E), \lambda)$ with lifetime τ, an integer $k \geq 2$, and a function $f : V \times [\tau] \to [k]$, we say that $e = \{u, v\} \in E$ is *properly colored* by f (or that f *properly colors* e) if there exists $t \in \lambda(e)$ such that $f(u, t) \neq f(v, t)$. We say that f is a *temporal k-coloring* of $\mathcal{G}$ if f properly colors every edge in E. Temporal 2-Coloring consists of deciding whether a temporal graph $\mathcal{G} = (G = (V, E), \lambda)$ admits a temporal 2-coloring (similarly, for any $k > 2$ we define Temporal k-Coloring). It is known that Temporal 2-Coloring is NP-complete [16] even if G has bounded treewidth.

Temporal Firefighter Reserve. Given a temporal graph $\mathcal{G} = (G = (V, E), \lambda)$ with lifetime τ, a *defence strategy (in $\mathcal{G}$)* is a sequence $\mathcal{S} = (S_1, \ldots, S_\tau)$ of pairwise disjoint subsets of V such that $|S_t| \leq t - \sum_{i=1}^{t-1} |S_i|$, for each $t \in [\tau]$. Given a vertex $s \in V$, the set $B_s(\mathcal{S}) = B_s^\tau(\mathcal{S})$ of *burnt nodes* by a *fire starting in s* is recursively defined as follows: $B_s^1(\mathcal{S}) = \{s\} \cup (N_1(s) \setminus S_1)$, and, for any $t \in [\tau]$ with $t > 1$, $B_s^t(\mathcal{S}) = B_s^{t-1}(\mathcal{S}) \cup (N_t(B_s^{t-1}(\mathcal{S})) \setminus \bigcup_{i=1}^{t} S_i)$. In words, starting in s, at each time step the fire spreads to the temporal neighbors of the current fire, except that defended vertices can never catch on fire (note that the condition on the cardinality of S_t is due to the fact that, at each time step t, it is possible to decide not to defend a new node and to use this saving for future time steps). Given a temporal graph $\mathcal{G} = (G = (V, E), \lambda)$, a vertex $s \in V$, and an integer $k \leq |V|$, the Temporal Firefighter Reserve problem consists of deciding whether there exists a defence strategy $\mathcal{S}$ such that $|B_s(\mathcal{S})| \leq |V| - k$ (that is, at least k vertices have not been burnt). It is known that Temporal Firefighter Reserve is NP-complete [12].

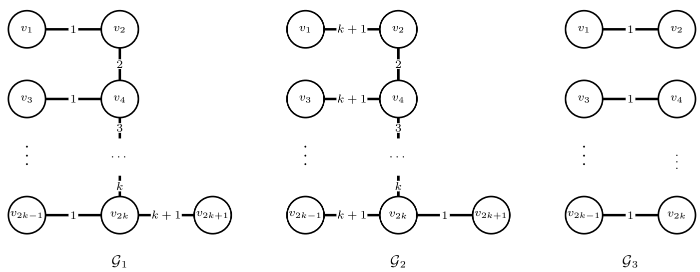

Fig. 1: For each edge e, the label of e denotes the unique time label in $\lambda(e)$. For any $k \geq 2$, we have that $\mathtt{imw}_{\leq}(\mathcal{G}_1) = 1$ since, for any $t \in [k+1]$, each connected component of $G_{\leq}(t)$ contains only one edge of Ψ^{e}_t. On the other hand, $\mathtt{imw}_{\geq}(\mathcal{G}_1) = k$ since $G_{\geq}(1) = \mathcal{G}_1$ is connected and Ψ^{e}_1 contains all the k edges $\{v_i, v_{i+1}\}$ for $i = \{1, 3, \ldots, 2k-1\}$. Similarly, we have that $\mathtt{imw}_{\leq}(\mathcal{G}_2) = k$ while $\mathtt{imw}_{\geq}(\mathcal{G}_2) = 1$. Finally, $\mathtt{imw}(\mathcal{G}_3) = k$ (since Ψ^{e}_1 contains all edges), while, for any $d \in \{\leq, \geq\}$, $\mathtt{imw}_d(\mathcal{G}_3) = 1$ (since each connected component of $G_d(1)$ contains only one edge).

Parameterized Complexity. We use standard notation and terminology from parameterized complexity theory [7] and we say that a problem is *fixed-parameter tractable* (FPT) with respect to a parameter k if it can be solved in time $f(k) \cdot n^{O(1)}$, where n is the size of the input.

2 Connected (vertex) interval membership width

Given a temporal graph $\mathcal{G} = (G = (V, E), \lambda)$ with lifetime τ, let the *active window* of an edge $e \in E$ be the interval $\mathtt{aw}(e) = [\min \lambda(e), \max \lambda(e)]$. Then, for each $t \in [\tau]$, let $\Psi^{\mathrm{e}}_t = \{f \in E : t \in \mathtt{aw}(f)\}$ be the *activity bag at time t* of $\mathcal{G}$. The *interval membership width* [3] of $\mathcal{G}$ is defined as $\mathtt{imw}(\mathcal{G}) = \max_{t \in [\tau]} |\Psi^{\mathrm{e}}_t|$. For each direction $d \in \{\leq, \geq\}$ and for each connected component C of $G_d(t)$, let $\Psi^{\mathrm{e}}_d(\mathcal{G}, t, C) = E(C) \cap \Psi^{\mathrm{e}}_t$ be the *d-connected bag* at time t of $\mathcal{G}$ and C. Moreover, let $\mathcal{F}^{\mathrm{e}}_d(t) = \{\Psi^{\mathrm{e}}_d(\mathcal{G}, t, C) : C$ is a connected component of $G_d(t)\}$ be the *family of d-connected bags* at time t of $\mathcal{G}$. The *d-connected interval membership width* of $\mathcal{G}$ is defined as $\mathtt{imw}_d(\mathcal{G}) = \max_{t \in [\tau], \Psi^{\mathrm{e}} \in \mathcal{F}^{\mathrm{e}}_d(t)} |\Psi^{\mathrm{e}}|$. Note that these parameters can be computed in (almost) linear time by using disjoint-set data structures [19] for implementing incremental connectivity algorithms.

We first observe that the $\leq$-connected interval membership width and the $\geq$-connected interval membership width are two incomparable measures. That is, there exists an infinite family of temporal graphs $\mathcal{G}_1$ for which $\mathtt{imw}_{\leq}(\mathcal{G}_1)$ is arbitrarily smaller than $\mathtt{imw}_{\geq}(\mathcal{G}_1)$, and there exists an infinite family of temporal graphs $\mathcal{G}_2$ for which $\mathtt{imw}_{\geq}(\mathcal{G}_2)$ is arbitrarily smaller than $\mathtt{imw}_{\leq}(\mathcal{G}_2)$ (see Fig. 1).

For any temporal graph $\mathcal{G}$ and for each $d \in \{\leq, \geq\}$, we have that $\mathtt{imw}_d(\mathcal{G})$ is at most $\mathtt{imw}(\mathcal{G})$, since each bag in $\mathcal{F}^{\mathrm{e}}_d(t)$ is a subset of Ψ^{e}_t. Moreover, the d-connected interval membership width can be arbitrarily smaller than the interval membership width (see the family of temporal graphs $\mathcal{G}_3$ shown in Fig. 1).

For any direction $d \in \{\leq, \geq\}$, we will also make use of the following vertex version of the d-connected interval membership width. Given a temporal graph $\mathcal{G} = (G = (V, E), \lambda)$ with lifetime τ, let the *active window* of a vertex $v \in V(G)$ be the interval $\mathtt{aw}(v) = [\min \bigcup_{u \in N_G(v)} \lambda(\{u, v\}), \max \bigcup_{u \in N_G(v)} \lambda(\{u, v\})]$. Then, for each $t \in [\tau]$, let $\Psi^{\mathrm{v}}_t = \{u \in V : t \in \mathtt{aw}(u)\}$ be the *activity vertex bag* at time t of $\mathcal{G}$. The *vertex interval membership width* [3] is defined as $\mathtt{vimw}(\mathcal{G}) = \max_{t \in [\tau]} |\Psi^{\mathrm{v}}_t|$. For each connected component C of $G_d(t)$, let $\Psi^{\mathrm{v}}_d(\mathcal{G}, t, C) = V(C) \cap \Psi^{\mathrm{v}}_t$ be the *vertex d-connected bag* at time t of $\mathcal{G}$ and C. Moreover, let $\mathcal{F}^{\mathrm{v}}_d(t) = \{\Psi^{\mathrm{v}}_d(\mathcal{G}, t, C) :\ C$ is a connected component of $G_d(t)\}$ be the *family of vertex d-connected bags* at time t of $\mathcal{G}$. The *d-connected vertex interval membership width* of $\mathcal{G}$ is defined as $\mathtt{vimw}_d(\mathcal{G}) = \max_{t \in [\tau], \Psi^{\mathrm{v}} \in \mathcal{F}^{\mathrm{v}}_d(t)} |\Psi^{\mathrm{v}}|$. Once again, for any direction d, the d-connected vertex interval membership width is not greater than the vertex interval membership width (since each bag in $\mathcal{F}^{\mathrm{v}}_d(t)$ is a subset of Ψ^{v}_t), and the $\leq$-connected vertex interval membership width and the $\geq$-connected vertex interval membership width are two incomparable measures, since the same examples given for the connected interval membership width work also for the connected vertex interval membership width.

3 Eulerian trails parameterized by $\mathtt{imw}_{\leq}$ and by $\mathtt{imw}_{\geq}$

In this section we consider the Temporal Eulerian Trail problem. This problem is FPT when parameterized by the interval membership width [3]. In this section, we show that this result can be improved by considering as a parameter the d-connected interval membership width, for any direction $d \in \{\leq, \geq\}$. Let us first consider the case $d = \leq$ (in the following, without loss of generality, we can assume that the underlying graph of the input temporal graph is connected). A (u, v, t)*-trailset* is a set of edges of $\mathcal{G}$ such that there exists a temporal trail from u to v in $\mathcal{G}$ which arrives in v at time at most t and which uses all and only the edges in the set. Given a temporal graph $\mathcal{G} = (G = (V, E), \lambda)$, let us define the following (dynamic programming) table $\overrightarrow{M}$. For each $t \in [\tau]$, each connected component C of $G_{\leq}(t)$, each $u, v \in V(C)$, and each subset F of $\Psi^{\mathrm{e}}_{\leq}(\mathcal{G}, t, C)$, we let $\overrightarrow{M}[t, C, u, v, F] = 1$ if and only if $E(C) \setminus F$ is a (u, v, t)-trailset.

Fact 1 *$\mathcal{G} = (G = (V, E), \lambda)$ has an Eulerian trail if and only if there exists u, v such that $\overrightarrow{M}[\tau, G, u, v, \emptyset] = 1$.*

Now, given $t \in [\tau]$, a connected component C of $G_{\leq}(t)$, a pair $u, v \in V(C)$, and $F \subseteq \Psi^{\mathrm{e}}_{\leq}(\mathcal{G}, t, C)$, we show how to recursively compute $\overrightarrow{M}[t, C, u, v, F]$.

Base case. If $t = 1$, then $\Psi^{\mathrm{e}}_1 = E(G_1)$ and $G_{\leq}(1) = G_1$: hence, $\Psi^{\mathrm{e}}_{\leq}(\mathcal{G}, 1, C) = E(C) \cap \Psi^{\mathrm{e}}_1 = E(C) \cap E(G_1) = E(C)$. We set $\overrightarrow{M}[1, C, u, v, F]$ to 1 if and only if $|F| = |E(C)| - 1$, $\{u, v\} \notin F$, and $1 \in \lambda(\{u, v\})$ (that is, $\{u, v\} \in E(C) \setminus F$).

Recursive step. Let $t \in [\tau]$ with $t > 1$. We set $\overrightarrow{M}[t, C, u, v, F]$ to 1 if and only if one of the following two cases occurs.

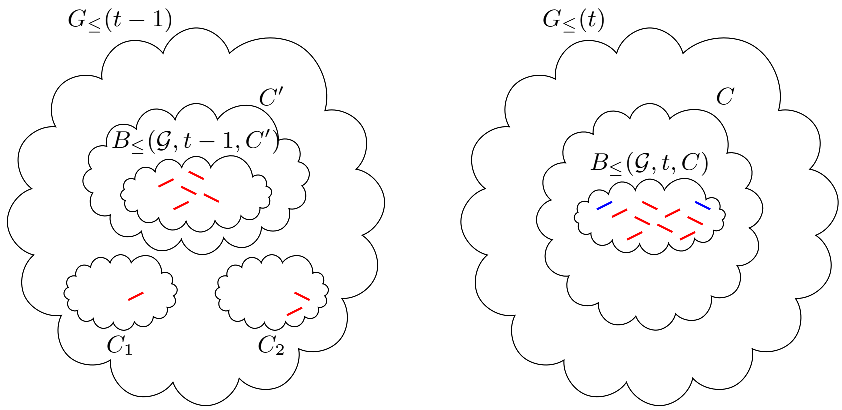

Fig. 2: Case 1 of the recursive step to compute table $\overrightarrow{M}$ for the Temporal Eulerian Trail problem. The red edges in $\Psi^{\mathrm{e}}_{\leq}(\mathcal{G}, t, C)$ are the edges in F which distribute among the three connected components of $G_{\leq}(t-1)$ in which the connected component C of $G_{\leq}(t)$ is split. The blue edges in $\Psi^{\mathrm{e}}_{\leq}(\mathcal{G}, t, C)$ are the edges in F which are not contained in $G_{\leq}(t-1)$. If the (u, v, t)-trailset in $G_{\leq}(t)$ is equal to $E(C')$ minus the red edges in $\Psi^{\mathrm{e}}_{\leq}(\mathcal{G}, t-1, C')$ (which form the set F'), then $\overrightarrow{M}[t, C, u, v, F]$ is set to 1 (the red edges of F in the other two connected components C_1 and C_2 are not included in F').

Case 1 There exists a connected component C' of $G_{\leq}(t-1)$ and a subset F' of $\Psi^{\mathrm{e}}_{\leq}(\mathcal{G}, t-1, C')$ such that $\overrightarrow{M}[t-1, C', u, v, F'] = 1$ and $E(C) \setminus F = E(C') \setminus F'$ (see Figure 2). In words, there is a desired trail finishing at time at most $t-1$.

Case 2 There exists $e = \{w, v\} \in E(C) \setminus F$ such that $t \in \lambda(e)$, a connected component C' of $G_{\leq}(t-1)$, and a subset F' of $\Psi^{\mathrm{e}}_{\leq}(\mathcal{G}, t-1, C')$ such that $\overrightarrow{M}[t-1, C', u, w, F'] = 1$ and $E(C) \setminus (F \cup \{\{w, v\}\}) = E(C') \setminus F'$. In words, the desired trail can be obtained through a trail from u to some w finishing at time at most $t-1$, and then using the edge $\{w, v\}$ active at time t.

By executing a "backward" version of the previously described dynamic programming algorithm, we can also solve Temporal Eulerian Trail parameterized by $\mathtt{imw}_{\geq}(\mathcal{G})$.

Theorem 2. *Given a temporal graph $\mathcal{G} = (G = (V, E), \lambda)$ with lifetime τ, the* Temporal Eulerian Trail *problem with input $\mathcal{G}$ can be solved in time $O(2^w \tau n^4 m)$ where $n = |V|$, $m = |E|$, and $w \in \{\mathtt{imw}_{\leq}(\mathcal{G}), \mathtt{imw}_{\geq}(\mathcal{G})\}$.*

4 Vertex coloring parameterized by $\mathtt{imw}_{\leq}$ and by $\mathtt{imw}_{\geq}$

In this section, we consider Temporal 2-Coloring. Deciding whether a temporal graph $\mathcal{G}$ admits a temporal 2-coloring is FPT when parameterized by the

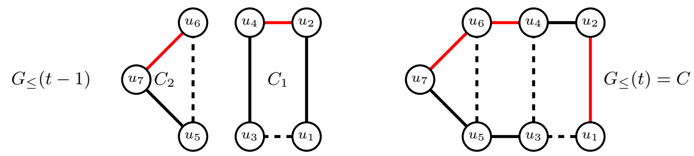

Fig. 3: The recursive step to compute table $\overrightarrow{M}$ for the vertex coloring problem. The dashed edges on the left (respectively, right) are the ones which are not in Ψ^{e}_{t-1} (respectively, Ψ^{e}_{t}). The red edges on the left (respectively, on the right) are the ones in F_1 and F_2 (respectively, F). In this case, $(E(C) \setminus F) \setminus ((E(C_1) \setminus F_1) \cup (E(C_2) \setminus F_2)) = \{\{u_3, u_5\}, \{u_2, u_4\}\}$. Since the graph induced by these two edges is bipartite, if $\overrightarrow{M}[t-1, C_1, F_1] = \overrightarrow{M}[t-1, C_2, F_2] = 1$, then we set $\overrightarrow{M}[t, C, F] = 1$.

treewidth of the underlying graph and the lifetime τ [16]. In the following, we show that this problem is FPT when parameterized by the d-connected interval membership width, for any direction $d \in \{\leq, \geq\}$. Let us first consider the case $d = \leq$. Given a temporal graph $\mathcal{G} = (G = (V, E), \lambda)$, let us define the following (dynamic programming) table $\overrightarrow{M}$. For each $t \in [\tau]$, each connected component C of $G_{\leq}(t)$ with edge set $E(C)$, and each subset F of $\Psi^{\mathrm{e}}_{\leq}(\mathcal{G}, t, C)$, we let $\overrightarrow{M}[t, C, F] = 1$ if and only if there exists a function $f : V(C) \times [t] \to [2]$ which properly colors every edge in $E(C) \setminus F$.

Fact 3 *Let $\mathcal{G} = (G = (V, E), \lambda)$ be a temporal graph with lifetime τ. There exists a temporal 2-coloring of $\mathcal{G}$ if and only if $\overrightarrow{M}[\tau, G, \emptyset] = 1$ for every connected component C of G.*

Now, given $t \in [\tau]$, a connected component C of $G_{\leq}(t)$ and a subset F of $\Psi^{\mathrm{e}}_{\leq}(\mathcal{G}, t, C)$, we show how to recursively compute $\overrightarrow{M}$.

Base case. If $t = 1$, then we set $\overrightarrow{M}[1, C, F]$ to 1 if and only if the graph $(V(C), E(C) \setminus F)$ is bipartite.

Recursive step. Let $t \in [\tau]$ with $t > 1$. We set $\overrightarrow{M}[t, C, F]$ to 1 if and only if there exist q connected components $C_1, \ldots, C_q$ of $G_{\leq}(t-1)$ and q sets $F_1, \ldots, F_q$ with $F_i \subseteq \Psi^{\mathrm{e}}_{\leq}(\mathcal{G}, t-1, C_i)$, for $i \in [q]$, such that the following two properties are satisfied (see Figure 3 for an example).

1. For every $i \in [q]$, $\overrightarrow{M}[t-1, C_i, F_i] = 1$.
2. The graph $(V(C), (E(C) \setminus F) \setminus \bigcup_{i=1}^{q} (E(C_i) \setminus F_i))$ is bipartite. Intuitively, this graph contains all the edges in $E(C) \setminus F$ which have not been properly colored before time t, and, hence, do not belong to any set $E(C_i) \setminus F_i$, for $i \in [q]$.

Again, by executing a "backward" version of the above algorithm, we can solve Temporal 2-Coloring problem parameterized by $\mathtt{imw}_{\geq}(\mathcal{G})$.

Theorem 4. *The* Temporal 2-Coloring *problem with input a temporal graph $\mathcal{G} = (G = (V, E), \lambda)$ with lifetime τ can be solved in time $O(4^w \tau n^2 m)$ where $n = |V|$, $m = |E|$, and $w \in \{\mathtt{imw}_{\leq}(\mathcal{G}), \mathtt{imw}_{\geq}(\mathcal{G})\}$.*

Observe, finally, that, in order to generalize our approach for higher values of k, it suffices to test whether $\Psi_d^{\mathrm{e}}(\mathcal{G}, 1, C) \setminus F = E(C) \setminus F$ is k-colorable in the base case, and whether f is a proper k-coloring of $(\Psi_d^{\mathrm{e}}(\mathcal{G}, t, C) \setminus F) \setminus \bigcup_{i=1}^{q}(E(C_i) \setminus F_i)$ in item 2 of the recursive step. This clearly increases the time complexity of the algorithm by factor which is exponential in the d-connected interval membership width, giving an FPT algorithm for the Temporal k-Coloring problem when parameterized by the d-connected interval membership width, for any direction $d \in \{\leq, \geq\}$, and k.

5 Firefighter parameterized by $\texttt{vimw}_{\leq}$

Temporal Firefighter Reserve is FPT when parameterized by the vertex interval membership width [12]. In the following, we show that this result can be improved by considering as a parameter the $\leq$-connected vertex interval membership width. Consider a temporal graph $\mathcal{G} = (G = (V, E), \lambda)$ with lifetime τ and a source node $s \in V(G)$. For each $t \in [\tau]$, we denote the connected component of $G_{\leq}(t)$ containing s by C_s^t and we denote by $\mathcal{G}_{\leq}^s(t)$ the related temporal graph (i.e., the temporal subgraph of $\mathcal{G}_{\leq}(t)$ constrained to C_s^t). Also, let n denote $|V(G)|$. Observe that if $A(s) = [t_1, t_2]$ and $t_1 > 1$, then C_s^t is the trivial component containing only s for every $t \in [t_1 - 1]$. In such case, we can apply the following procedure starting at t_1 instead. To make presentation simpler, in what follows we consider that $t_1 = 1$. Let us now define the following (dynamic programming) table $\overrightarrow{M}$. For each pair $d, t \in [\tau]$, each two subsets D and B of $\Psi_{\leq}^{\mathrm{v}}(\mathcal{G}, t, C_s^t)$, and each $b \in [n]$, we let $\overrightarrow{M}[t, D, B, d, b] = 1$ if and only if there exists a defence strategy $\mathcal{S} = (S_1, \ldots, S_t)$ in $\mathcal{G}_{\leq}^s(t)$ such that:

1. D is the set of vertices defended in bag t. Formally $D = \Psi_{\leq}^{\mathrm{v}}(\mathcal{G}, t, C_s^t) \cap \bigcup_{i=1}^{t} S_i$;
2. B is the set of burnt vertices in bag t. Formally $B = \Psi_{\leq}^{\mathrm{v}}(\mathcal{G}, t, C_s^t) \cap B_s^t(\mathcal{S})$;
3. d is the total number of defended vertices up to time t. Formally, $d = \sum_{i=1}^{t} |S_i|$. Observe that $d \leq t$ by definition; and
4. b is the total number of burnt vertices at time t. Formally, $b = |B_s^t(\mathcal{S})|$.

Fact 5 *$(\mathcal{G}, s, k)$ is a yes-instance of the* Temporal Firefighter Reserve *problem if and only if there exists D, B, d, b such that $\overrightarrow{M}[\tau, D, B, d, b] = 1$ and $n - b \geq k$.*

Now, given $d, t \in [\tau]$, two subsets D and B of $\Psi_{\leq}^{\mathrm{v}}(\mathcal{G}, t, C_s^t)$, and $b \in [n]$, we show how to recursively compute $\overrightarrow{M}[t, D, B, d, b]$.

Base case. Let $t = 1$. If $D = \emptyset$ (i.e., no vertex is defended at time 1), then we set $\overrightarrow{M}[1, D, B, d, b]$ to 1 if $B = \{s\} \cup N_1(s)$, $d = 0$ and $b = |B|$. If $D = \{u\}$ for some $u \in V(C_s^1)$, then we set $\overrightarrow{M}[1, D, B, d, b]$ to 1 if $B = \{s\} \cup (N_1(s) \setminus \{u\})$, $d = 1$ and $b = |B|$. In all other cases, we set $\overrightarrow{M}[1, D, B, d, b]$ to 0.

Recursive step. Let $t \in [\tau]$ with $t > 1$. We set $\overrightarrow{M}[t, D, F, d, b]$ to 1 if and only if there exist D', B', d', b' such that $\overrightarrow{M}[t-1, D', B', d', b'] = 1$ and the following properties are satisfied.
- (I) $D = (D' \cap \Psi^{\mathrm{v}}_{\leq}(\mathcal{G}, t, C^t_s)) \cup A$, where $A \subseteq \Psi^{\mathrm{v}}_{\leq}(\mathcal{G}, t, C^t_s)) \setminus (B' \cup D')$.
- (II) $B = (B' \cap \Psi^{\bar{\mathrm{v}}}_{\leq}(\mathcal{G}, t, C^t_s)) \cup (N_t(B') \setminus D)$.
- (III) $d = d' + |A|$.
- (IV) $b = b' + |N_t(B') \setminus D|$.

Theorem 6. *The* Temporal Firefighter Reserve *problem, with input a temporal graph $\mathcal{G} = (G = (V, E), \lambda)$ with lifetime τ, $s \in V$, and $k \in [n]$, can be solved in time in time $O(16^w \tau^4 n^2)$, where $n = |V|$ and $w = \mathtt{vimw}_{\leq}(\mathcal{G})$.*

6 Bidirectional connected interval membership width

Given a temporal graph $\mathcal{G} = (G = (V, E), \lambda)$ with lifetime τ, for each $t \in [\tau]$, we define the *bidirectional connected interval membership width* at time t as

$$\mathtt{imw}_{\sim}(t) = \begin{cases} \max\{\mathtt{imw}_{\leq}(\mathcal{G}_{\leq}(t-1)), \mathtt{imw}_{\geq}(\mathcal{G}_{\geq}(t+1)), |\Psi^{\mathrm{e}}_t|\} & \text{, if } 1 < t < \tau \\ \mathtt{imw}_{\geq}(\mathcal{G}) & \text{, if } t = 1 \\ \mathtt{imw}_{\leq}(\mathcal{G}) & \text{, if } t = \tau. \end{cases}$$

The *bidirectional connected interval membership width* of $\mathcal{G}$ is then defined as $\mathtt{imw}_{\sim}(\mathcal{G}) = \min_{t \in [\tau]} \mathtt{imw}_{\sim}(t)$. Note that also this parameter can be computed in (almost) linear time and that there exists an infinite family of temporal graphs $\mathcal{G}$ for which $\mathtt{imw}_{\sim}(\mathcal{G})$ is arbitrarily smaller than $\mathtt{imw}_d(\mathcal{G})$, for any direction d (see Fig. 4). In order to solve a problem by referring to the bidirectional connected interval membership width, we can first solve it with input $\mathcal{G}_{\leq}(t-1)$ and with input $\mathcal{G}_{\geq}(t+1)$, by referring to the connected interval membership width of $\mathcal{G}_{\leq}(t-1)$ with direction $\leq$ and the connected interval membership width of $\mathcal{G}_{\geq}(t+1)$ with direction $\geq$, and then combine the two solutions. This approach can be followed in the case of the Temporal Eulerian Trail problem and of the Temporal 2-Coloring problem, as stated by the following two results.

Theorem 7. *Given a temporal graph $\mathcal{G} = (G, \lambda)$ with n nodes, m edges, and lifetime τ, the* Temporal Eulerian Trail *problem can be solved by an algorithm running in time $O(2^{\mathtt{imw}_{\sim}(\mathcal{G})} \tau n^4 m)$.*

Theorem 8. *Given a temporal graph $\mathcal{G} = (G, \lambda)$ with n nodes, m edges, and lifetime τ, the* Temporal 2-Coloring *problem can be solved by an algorithm running in time $O(4^{\mathtt{imw}_{\sim}(\mathcal{G})} \tau n^2 m)$.*

7 Conclusion

We have introduced a three-level hierarchy of polynomial-time computable parameters starting from the interval membership width up to the bi-directional

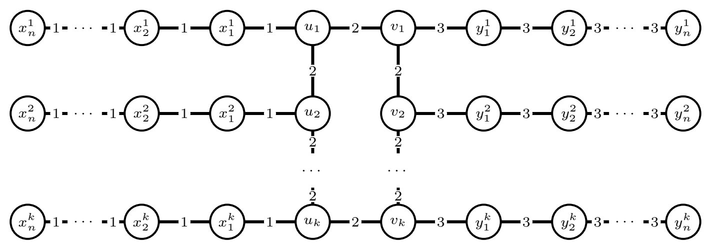

Fig. 4: For each edge e, the label of e denotes the unique time label in $\lambda(e)$. For each $k, n \geq 1$ and for any direction d, we have that $\mathtt{imw}_d(\mathcal{G}) = (k+1)n$ (since $|\Psi_1^{\mathrm{e}}| = |\Psi_3^{\mathrm{e}}| = (k+1)n$, $G_{\leq}(3) = G_{\geq}(1) = G$, and G is connected), while $\mathtt{imw}_{\sim}(\mathcal{G}) = n$ (since $|\Psi_2^{\mathrm{e}}| = 2(k+1)$ and $\mathtt{imw}_{\leq}(\mathcal{G}_{\leq}(1)) = \mathtt{imw}_{\geq}(\mathcal{G}_{\geq}(3)) = n$).

version of the connected interval membership width (see Fig. 5). We proved that the hierarchy is strict (that is, each parameter at one level can be arbitrarily smaller than the ones in the levels below), and that the two parameters at the second level are not comparable (that is, each of them can be arbitrarily smaller than the other). We also proved that Temporal Eulerian Trail and Temporal 2-Coloring are FPT with respect to any of the parameters in the hierarchy, and that Temporal Firefighter Reserve is FPT with respect to the vertex-variant of the parameter at the first level and of one of the parameters of the second level. A natural research direction will be to design FPT algorithms with respect to our new parameters for other temporal graphs problems (such as counting temporal paths [8]). We also leave as an open question the classification of Temporal Firefighter Reserve when parameterized by $\mathtt{vimw}_{\geq}(\mathcal{G})$.

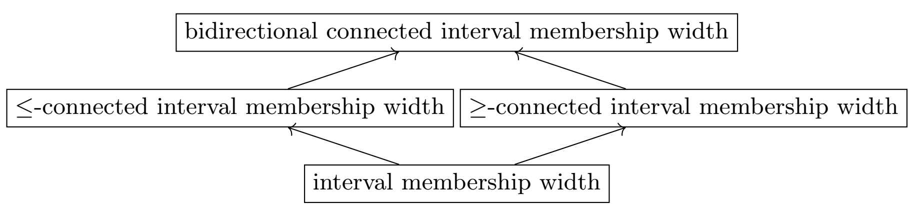

Fig. 5: The hierarchy of interval membership width parameters.

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