On Families of Limit S-adic Words

Gwenaël Richomme

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LIRMM, Université Paul-Valéry Montpellier 3, Université de Montpellier, CNRS, Montpellier, France gwenael.richomme@lirmm.fr
http://www.lirmm.fr/~richomme/

Abstract. Given a set S of morphisms, an infinite word is limit S-adic if it can be recursively desubstituted using morphisms in S. Substitutive-adicity arises naturally in various studies especially in studies on infinite words with factor complexity bounded by an affine function. In the literature, when a family F of infinite words defined by a combinatorial property P appears to be S-adic for some set S of morphisms, it is very rare that the whole set of limit S-adic words coincides with F. The aim of the talk is to survey such situations in which necessarily morphisms of S preserve the property P of infinite words.

Keywords: S-adicity · Property preserving morphisms.

We assume that readers are familiar with combinatorics on words; for omitted definitions see, *e.g.*, [11,12,5]. All the infinite words considered in this abstract are right infinite words.

As explained with more details in [3], the terminology S-adic was introduced by S. Ferenczi [8]. Without context, letter S refers to term "substitution" (and we will sometimes use the terminology *substitutive-adicity* instead) and in more precise definitions, it refers to a set S of (nonerasing) morphisms. An infinite word $\mathbf{w}$ is said S*-adic* if there exist a sequence $(f_n)_{n\geq 1}$ of morphisms in S and a sequence of letters $(a_n)_{n\geq 1}$ such that $\lim_{n\to+\infty} |f_1 f_2 \cdots f_n(a_{n+1})| = +\infty$ and $\mathbf{w} = \lim_{n\to+\infty} f_1 f_2 \cdots f_n(a_{n+1}^\omega)$. The sequence $(f_n)_{n\geq 1}$ is called a *directive word* of $\mathbf{w}$. Assume limits $\mathbf{w}_k = \lim_{n\to+\infty} f_k f_{k+1} \cdots f_n(a_{n+1}^\omega)$ exist for all k. Observe that $\mathbf{w}_1 = \mathbf{w}$ and $\mathbf{w}_n = f_n(\mathbf{w}_{n+1})$ for all $n \geq 1$, that is, $\mathbf{w}$ can be infinitely desubstituted using morphisms in S. Following [1] (where is used terminology *limit point* of a sequence of substitutions), we say that $\mathbf{w}$ is a *limit S-adic* word when such sequences $(f_n)_{n\geq 1}$ and $(\mathbf{w})_{n\geq 1}$ exist. Observe that the set of limit S-adic words is the minimal set of infinite words X such that $X = \bigcup_{f\in S} f(X)$ that we could denote by abuse of notation $X = S(X)$ to emphasize the fact that limit S-adic words are generalizations of fixed points of morphisms (and even of morphic sequences).

Limit substitutive-adicity arises naturally in various studies as, for instance, this of Sturmian words (see, *e.g.* [12, Chap.2] and [13, Chap. 5]). It is well-known (and easy to prove) that any Sturmian word is a limit S_{sturm}-adic word with $S_{sturm} = S_a \cup S_b$, $S_a = \{L_a, R_a\}$, $S_b = \{L_b, R_b\}$, $L_a(a) = a = R_a(a)$, $L_a(b) = ab$, $R_a(b) = ba$, $L_b(a) = ba$, $R_b(a) = ab$, $L_b(b) = b = R_b(b)$. But not all limit S_{sturm}-adic words are Sturmian. Only the limit S_{sturm}-adic words whose directive word contains infinitely many elements of S_a and infinitely many elements of S_b are Sturmian. This condition can be described using infinite paths with prohibited segments in an automaton (or a graph), here with two states (or vertices), one for each set S_a and S_b. This kind of condition with an automaton to characterize allowed directive words is also used, for instance, in the characterization of words for which the first difference of factor complexity is bounded by 2 [10] or in the characterization of sequences arising from the study of multidimensional continued fraction algorithm (see for instance [4,6]).

In [14], answering a question of G. Fici, the author characterizes in term of limit S-adicity the family of so-called LSP infinite words, that is the words having all their left special factors as prefixes. For this he determines a suitable set S_{bLSP} of morphisms and an automaton recognizing allowed infinite desubstitutions. As the obtained characterization is quite evolved, a second part of [14] considers the question of finding a simpler S-adic characterization. It is proved that, unfortunately, it does not exist any set of morphisms S such that the family of LSP infinite words is (exactly) the family of limit S-adic words except in the binary case.

The aim of the talk is to consider the question : which are the known families of infinite words defined by a combinatorial property P that correspond to a family of limit S-adic words for some set of S of morphisms? In [14], it was observed that when such a situation arises necessarily morphisms of S preserve the property P of infinite words. In what follows we will say that a morphism preserves words of a family F if it maps any word on this family on another word of this family.

In the case of Sturmian words, morphisms in S_{sturm} indeed preserve Sturmian words. That all S_{sturm}-adic infinite words are not Sturmian comes from the fact that some limit S_{sturm}-adic words are periodic. At this stage, readers should remember that Sturmian words are the aperiodic binary balanced words (where balanced means that the numbers of a occurring in any factors u and v of equal length may differ by at most one). Morphisms in S_{sturm} preserve binary balanced words and consequently limit S_{sturm}-adic words are the infinite binary balanced words.

Actually it is easy to observe that the family of Sturmian words corresponds to a family of limit substitutive-adic words, more precisely to the family of limit S'_{sturm}-adic words where $S'_{sturm} = \bigcup_{n\geq 1} S_a^n S_b \cup S_b^n S_a$. Morphisms of S'_{sturm} are obtained composing morphisms to enforce infinite occurrences of morphisms of each of the sets S_a and S_b in directive words of Sturmian words. One can note that in this case the set of morphisms S'_{sturm} is infinite.

A similar situation holds for the Arnoux-Rauzy words [2]. They are limit-S_E-adic words for a finite set of morphisms that generalize Sturmian words and they correspond to a family of limit substitutive-adic words for an infinite set of morphisms (obtained concatenating morphisms in S_E). The morphisms in S_E^* are morphisms that preserve episturmian words [7,9] and the set of episturmian words is exactly the set of S_E-adic words.

Although limit S-adicity reveals itself to be a useful tool to study some combinatorial properties, although the notion of property preserving morphisms is often considered to generate words with interesting properties, the family of right infinite balanced words and the family of episturmian words seems to be the unique known families of words that correspond exactly to a family of limit S-adic words with S a *finite* set of morphisms.

Acknowledgment

Many thanks to J. Leroy and P. Séébold for their remarks on an earlier version of this summary.

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