On Maxitive Image Processing

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Abstract Digital image processing has become the most common form of image processing. Many transformations can be achieved by very simple and versatile algorithms such as contrast enhancing, restoration, color correction, etc. However, a wide branch of image processing algorithms make an extensive use of spatial transformations that are only defined in the analog domain such as rotation, translation, zoom, anamorphosis, homography, distortion, derivation, etc. Designing a digital image processing algorithm that mimic a spatial transformation is usually achieved by using the so-called *kernel based approach*. This approach involves two kernels to ensure the continuous to discrete interplay: the sampling kernel and the reconstruction kernel, whose choice is highly arbitrarily made. The maxitive kernel based approach can be seen as an extension of the conventional kernel based approach that reduces the impact of such an arbitrary choice. Replacing a conventional kernel by a maxitive kernel in a digital image spatial transformation leads to compute the convex set of all the images that would have been obtained by using a (continuous convex set) of conventional kernels. Using this set induces a kind of robustness that can reduce the risk of false interpretation. Medical imaging for example would be a kind of applications that could benefit of such an approach.

1 Introduction

In digital image processing, fuzzy subsets have been used from its very first introduction for representing image information at different levels (seen e.g. [2] for

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a nice overview on fuzzy set based image processing). Digital image processing refers to the set of algorithms used to transform, filter, enhance, modify, analyze, distort, fuse, etc., digital images. Many of these algorithms are designed to mimic an underlying physical operation defined in the continuous illumination domain and formerly achieved via optical or electronic filters or through manipulations, including painting, cutting, moving or pasting of image patches. Spatial domain operations like derivation [22], morphing, filtering, geometric and perspective transformations [12], super-resolution [16], etc. are usually derived by using a kernel based approach [23]. In kernel-based approaches, the choice of a particular kernel shape or spread is usually prompted more by practical aspects than by any theoretical purpose. Unfortunately, this choice can highly impact the output of the obtained discrete operator. Figure 2 illustrates this fact by highlighting the noise introduced by a digital approximation of a continuous rigid transformation operation. Two different interpolation methods have been used to rotate a detail of Figure 1 (see Figures 2.a,b). The obtained images are not identical as illustrated by enhancing their absolute difference (Figure 2.c).

This dependance is not a real problem when the considered operations are dedicated to artistic modifications of an image. Photographs have their own rule to choose among the three main interpolation methods (nearest neighbor, bilinear, bicubic), while some dedicated softwares have developed their own interpolation (or more generally reconstruction) method. It is more problematic when the information carried on by an image is quantitative, e.g. in medical applications where quantization is expected.

Fig. 1 A digitalized version of an illustration of Ivan Bilibin for Fairytale of the Tsar Saltan (1905).

For example to study a lung tumor growth, the patient is subjected to hybrid PET-CT scan, where the CT gives the anatomical structure information and PET gives quantitative information about the tumor metabolism. After image acquisition, it is necessary to bring the image of either of one the modality w.r.t to other (PET w.r.t CT or vice versa). This operation involves geometrical rigid or non rigid transformations. If details in the images are comparable to the image resolution, the choice of registration algorithm can be critical. As an example, on Figure 3, a small ganglion that is a negative marker of recovery could be ignored.

A more careful approach would be to compute not a single transformed image but the set of all images that could have been obtained by using different kernels. In medical diagnosis, this can lead to confirm a diagnosis (if all the images lead to the same diagnosis) or highlight the need for a complementary medical investigation (if different images lead to different diagnosis). This is what the maxitive approach proposes.

Fig. 2 Rotation of 3° of a detail of Figure 1 by using bilinear interpolation (a) bicubic interpolation (b) and an enhanced view of the difference between the two rotated images (c)

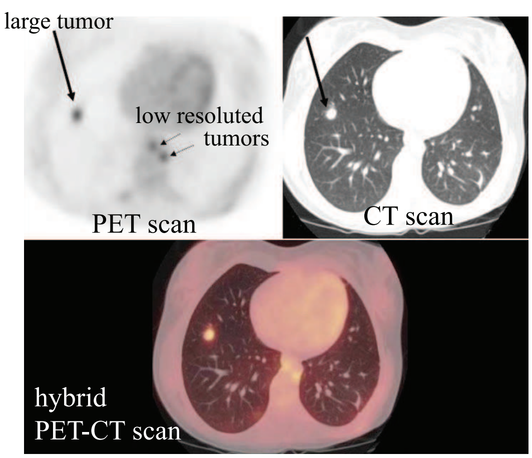

Fig. 3 The large tumor is visible on the PET scan but the low resoluted tumors are fuzzy. They are almost not visible on the hybrid PET-CT scan, due to a not appropriate choice of kernel during the registration.

Maxitive image processing takes advantage of the obvious analogy between probability density functions (pdf) and positive kernels to extend the signal processing theory to the case where the modeling is imprecisely known [13]. Within this technique, the possibilistic interpretation [5] of fuzzy subsets is used to define *maxitive kernels* that can be seen as convex sets of conventional positive kernels. These convex sets aim at representing scant knowledge on the appropriate kernel to be used in a given application. Maxitive-based signal processing extensions lead to interval-valued signal that includes the set of all signals that would have been obtained by the corresponding conventional method using a positive kernel that belongs to the core of the maxitive kernel [17]. This approach has been extended into two dimensions and involved in image processing applications (see e.g. [9, 6, 19, 10]).

In this article, we propose a formalization of the maxitive approach for extending geometrical transformations in digital image processing, i.e. we show how this technique can be used to compute the convex set of all images that would have been obtained by considering a convex set of possible kernels when applying geometrical transformations to digital images.

After this introduction, Section 2 provides some notations and some necessary background knowledge. In Section 3.2 we present the kernel-based method to design discrete operators mimicking continuous geometrical transformations. We then propose the maxitive approach as a simple extension of the former method. We propose to use this extension to design a rigid transformation. We then conclude this article.

2 Preliminary considerations, definitions and notations

2.1 Notations

Let $\mathbb{R}$ be the real line and $\mathbb{IR}$ be the set of all intervals of $\mathbb{R}$. Let Ω be the image plane, i.e. a box of $\mathbb{R}^2$: $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1, \Omega_2 \subseteq \mathbb{R}$. Let $\mathscr{P}(\Omega)$ be the set of all Lebesgue measurable subsets of Ω. With N being a positive integer, we define Θ_N by $\Theta_N = \{1, \ldots, N\} \subset \mathbb{N}$. Let $\mathscr{P}(\Theta_N) = 2^{\Theta_N}$ be the power set of Θ_N.

2.2 Capacities and expectations

A capacity is a confidence measure that is more general than a probability measure. It can be defined on both continuous and discrete domains. A capacity defined on a continuous reference set is called a *continuous capacity*, while a capacity defined on a discrete reference set is called a *discrete capacity*. Let Φ be either Ω or Θ_N.

Definition 1. *A (continuous or discrete) capacity ν is a set function $\nu : \mathscr{P}(\Phi) \to [0,1]$ such that $\nu(\varnothing) = 0$, $\nu(\Phi) = 1$ and $\forall A, B \in \mathscr{P}(\Phi)$, $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$.*

Given a capacity ν, its conjugate ν^c is defined as: $\nu^c(A) = 1 - \nu(A^c)$ for any subset $A \in \mathscr{P}(\Phi)$, with A^c being the complementary set of A in Φ. A capacity ν such that for all A, B in $\mathscr{P}(\Phi)$, $\nu(A \cup B) + \nu(A \cap B) \leq \nu(A) + \nu(B)$ is said to be concave. Here we only consider this kind of capacity. The core of a concave capacity ν, denoted $\mathscr{M}(\nu)$, is the set of probabilities P on $\mathscr{P}(\Phi)$ such that $\nu(A) \geq P(A)$ for all subsets $A \in \mathscr{P}(\Phi)$. A probability measure is a capacity that equals its conjugate. Thus, if ν is a probability, $\nu(A \cup B) + \nu(A \cap B) = \nu(A) + \nu(B)$ [4].

The concept of expected value associated with a probability measure has been extended to concave capacities by means of a Choquet integral (see e.g. [18]). Let ν be a concave capacity defined on Φ and let $f : \Phi \to \mathbb{R}$ be a L_1 bounded function. The (imprecise) expectation of f w.r.t. ν is the real interval $\overline{\underline{\mathbb{E}}}_\nu(f)$ defined by:

$$\overline{\underline{\mathbb{E}}}_\nu(f) = \left[\underline{\mathbb{E}}_\nu(f), \overline{\mathbb{E}}_\nu(f)\right] = \left[\check{\mathbb{C}}_{\nu^c}(f), \check{\mathbb{C}}_\nu(f)\right],$$

where $\check{\mathbb{C}}$ denotes the asymmetric Choquet integral (see e.g. [21]). One of the important properties of this extension is that $\overline{\underline{\mathbb{E}}}_\nu(f)$ is an interval that contains all of the $\mathbb{E}_P(f)$ with $P \in \mathscr{M}(\nu)$. Conversely, any value of this interval corresponds to an expected value of the form $\mathbb{E}_P(f)$ with $P \in \mathscr{M}(\nu)$ [4].

Remark 1. *Upper expectations and concave capacities coincide when considering the characteristic function. Let ν be a concave capacity, $\forall A \in \mathscr{P}(\Phi)$, $\nu(A) = \overline{\mathbb{E}}_\nu(\chi_A)$, with χ_A being the characteristic function of A.*

2.3 Summative and maxitive kernels

In discrete image processing, the role of a kernel is to define a weighted neighborhood of spatial locations in the image plane, with those weights being used in an aggregation process. Among different kernels, summative kernels play an important role since they define normalized positive weighted neighborhoods. They are intensively used to establish discrete operators defined in the continuous domain [14]. Let $N \in \mathbb{N}$ and Φ be either Ω or Θ_N.

A **summative kernel** is a positive function $\kappa : \Phi \longrightarrow \mathbb{R}^+$ complying with the summative property, i.e. $\int_\Omega \kappa(x)dx = 1$ (if $\Phi = \Omega$) or $\sum_{n \in \Theta_N} \kappa_n = 1$(if $\Phi = \Theta_N$). Such a function defines a probability measure P_κ on Φ by: $\forall A \in \mathscr{P}(\Phi)$, $P_\kappa(A) = \int_A \kappa(x)dx$ (if $\Phi = \Omega$) or $P_\kappa(A) = \sum_{n \in A} \kappa_n$ (if $\Phi = \Theta_N$). $\mathscr{K}(\Phi)$ is the set of all summative kernels defined on Φ.

A **maxitive kernel** [14] is a function $\pi : \Phi \longrightarrow [0,1]$ complying with the maxitive property, i.e. $\sup_{x \in \Phi} \pi(x) = 1$. Such a function defines a concave capacity on Φ called a possibility measure Π_π by: $\Pi_\pi(A) = \sup_{x \in A} \pi(x)$. A maxitive kernel π defines a convex subset of $\mathscr{K}(\Phi)$ as follows: $\mathscr{M}(\pi) = \{\kappa \in \mathscr{K}(\Phi) | \forall A \in \mathscr{P}(\Phi), P_\kappa(A) \leq \Pi_\pi(A)\}$ called its *core*.

2.4 Crisp and fuzzy partitions

In image processing, partitioning is mandatory to define the relation between the continuous domain, where the illumination function is defined, and the discrete domain, where the measured illumination is depicted. Traditional image processing is based on crisp partition, while more advanced image processing has been based on fuzzy partition (see e.g. [15, 6]).

An image partition of Ω is a set of N subsets $\{C_n\}_{n\in\Theta_N}$ such that (i) $\forall(n,m)\in\Theta_N$, $C_n\cap C_m\neq\varnothing\Longleftrightarrow n=m$, (ii) $\forall\omega\in\Omega$, $\exists n\in\Theta_N$ such that $\omega\in C_n$.

An image partition is said to be uniform if it can be generated by a simple generic subset E: let χ_E be the characteristic function of E, $\forall n\in\Theta_N$, $\exists\omega_n\in\Omega$ such that $\forall\omega\in\Omega$, $\chi_{C_n}(\omega)=\chi_E(\omega-\omega_n)$.

A fuzzy image partition of Ω is a set of N fuzzy subsets $\{C_n\}_{n\in\Theta_N}$ such that $\forall\omega\in\Omega$: (i) $\sum_{n=1}^{N}\mu_{C_n}(\omega)=1$ and (ii) μ_{C_n} is continuous, with μ_{C_n} being the membership function of C_n ($n\in\Theta_N$). A fuzzy partition is said to be uniform if it can be generated by a simple generic fuzzy subset E: let $\{\omega_n\}_{n\in\Theta_N}$ be a set of N regularly spaced locations of Ω, then C_n is generated by E i.e. $\forall\omega\in\Omega$, $\mu_{C_n}(\omega)=\mu_E(\omega_n-\omega)$. A fuzzy partition is said to be normalized if $\forall n\in\Theta_N$, $\exists\omega\in\Omega$ such that $\mu_{C_n}(\omega)=1$ [20]. Usually, in image processing, partitions are uniform to comply with the geometry of the image sensors.

Fuzzy partitions are instrumental for performing reconstructions. Let $\{F_n\}_{n\in\Theta_N}$ be a discrete function, a reconstructed continuous function $\hat{F}:\Omega\to\mathbb{R}$ can be defined by $\forall\omega\in\Omega$, $\hat{F}(\omega)=\sum_{n\in\Theta_N}F_n\mu_{C_n}(\omega)$. The following definition will allow us to extend this instrumentality to link the continuous space Ω to the discrete space Θ_N.

Definition 2. *Let $A\subseteq\Theta_N$, then we define Υ_A as being the membership function of $\bigcup_{n\in A}C_n$, where the union is defined by the Łukasiewicz T-conorm: $\forall\omega\in\Omega$, $\Upsilon_A(\omega)=min(1,\sum_{n\in A}\mu_{C_n}(\omega))=\sum_{n\in A}\mu_{C_n}(\omega)$ due to Property (i).*

Remark 2. *Note that a uniform crisp partition is a special case of fuzzy partition where the generic fuzzy subset E is a crisp subset.*

3 From continuous to digital image processing

3.1 Continuous image / digital image

In the continuous domain, an image can be seen as a measurable physical illumination phenomenon, i.e. the projection, via an optical device, of the real-world light information in a particular direction. It is generally modeled by a bounded positive integrable function $\mathscr{I}$ defined on $\mathbb{R}^2$. More precisely $\mathscr{I}$ is a $L_1(\mathbb{R}^2)^+$ function defined on a compact subset Ω (e.g. a closed rectangle) of $\mathbb{R}^2$. This function is usually extended throughout the continuous domain $\mathbb{R}^2$ by assigning an arbitrary value γ (usually 0) to Ω^c (the complementary set of Ω in $\mathbb{R}^2$).

There are some optical systems that allow to perform image processing in the continuous domain. But nowadays image processing is mainly performed on computer or smartphones, i.e. on image stored in computer memory is a discrete quantity. From a signal processing point of view, a sampled image can be considered as being obtained by measuring the continuous illumination function $\mathscr{I}$ defined on $\mathbb{R}^2$ projected by an optical device on a matrix of sensors called the retina (see Figure 4). The sensors are usually regularly spaced along each axis at a limited number N of locations, called the *sampling locations*. Those measurement values or locations are usually referred to as *pixel values or locations*. Let $\Theta_N = \{1, \ldots, N\} \subset \mathbb{N}$ be the set of indices of the sampling locations and $\{\omega_n\}_{n \in \Theta_N}$ be the set of sampling locations also referred to as the *sampling grid*.

Ideally, each measure I_n can be modeled by integrating the illumination function $\mathscr{I}$ in a crisp neighborhood around ω_n. Let $\phi^{\omega_n} \subset \mathbb{R}^2$ be this neighborhood, then the relation between I and $\mathscr{I}$ can be expressed by: $\forall n \in \Theta_N,\ I_n = \int_{\phi^{\omega_n}} \mathscr{I}(\omega) d\omega$. When $\chi_{\phi^{\omega_n}}$ is the characteristic function of the subset ϕ^{ω_n}, it can be rewritten as:

$$\forall n \in \Theta_N,\ I_n = \int_{\mathbb{R}^2} \mathscr{I}(\omega) \chi_{\phi^{\omega_n}}(\omega) d\omega. \tag{1}$$

Finally, the measured pixel values are quantized to obtain the digital image. What has to be kept in mind is that what we have at hand is not the image but discrete measures of it.

There are many operations for which accounting for the underlying continuous nature of the image is mandatory: derivation, morphing, filtering, geometric and perspective transformations, etc. For those operations, the aim is to define a discrete operator that can mimic the equivalent operation in the continuous domain. The idea is illustrated in Figure 5 when considering a rotation of a detail of the image depicted in Figure 1 around the optical axis. Let us consider the input discrete

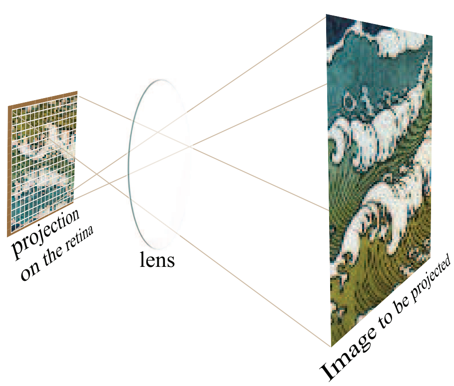

Fig. 4 Measure of a continuous image.

image as being obtained by sampling a continuous image. The image we would like to obtain by using the discrete rotation operator is the image that would have been obtained by rotating and then sampling the original continuous image. Such an operation is not possible due to the loss of information induced by the sampling. It has to be approximated in a way that preserves at best the original (discrete) information. For example, a particularly desirable property would be the reversibility of a digital operation. However, continuous based discrete operations always lead to information loss [3]. Therefore, the original information cannot be reconstructed from the processed image.

3.2 *Kernel-based image processing*

Kernel-based image processing, as illustrated in Figure 5, consists in defining discrete operations on digital images that are analog to operations defined on continuous images in the continuous domain. Kernels are used for defining weighted neighborhoods of a location in the image plane, aiming at reconstructing a continuous image from a discrete image, or sampling a continuous image to built a discrete image.

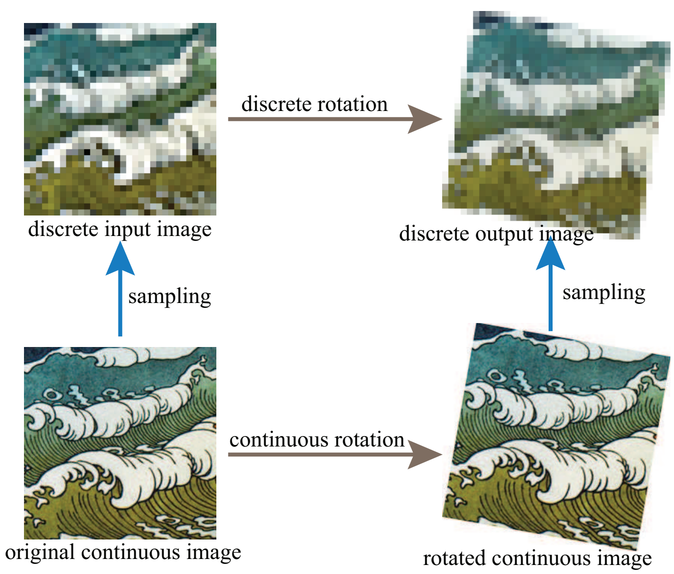

Fig. 5 How to go from continuous to discrete image processing?

3.2.1 Kernel-based image sampling and reconstruction

Let $\{\omega_n\}_{n\in\Theta_N}$ be the N sampling locations. As a very straightforward modeling, sampling a continuous image can be seen as the integration of the illumination function $\mathscr{I}$ in a neighborhood around each sampling location ω_n.

Let κ be a sampling kernel, the link between the sampled pixel values I and the continuous illumination $\mathscr{I}$ can be written as [7]:

$$\forall n \in \Theta_N,\ I_n = \int_{\mathbb{R}^2} \mathscr{I}(\omega)\kappa(\omega - \omega_n)d\omega. \tag{2}$$

A certain consistency has to be kept between the continuous and the discrete domain. Since the scaling of intensity values induced by quantization is usually unknown, the original illumination measurement scale is usually replaced by the available grayscale. Thus, we need to assume that the original image can be expressed in the digital grayscale. The consistency of both continuous and discrete images can be expressed in that way: "if $\mathscr{I}$ is a constant image such that $\forall \omega \in \Omega$, $\mathscr{I}(\omega) = a$, then $\forall n \in \Theta_N$, $I_n = a$". This implies that κ is a summative kernel.

Reconstruction can be thought of as a converse procedure. However, since sampling induces information loss, the recomposed image usually cannot be seen as a perfect reconstruction of the original continuous illumination $\mathscr{I}$, but rather as an estimate $\hat{\mathscr{I}}$ of the continuous function in Ω. This estimate is obtained by a finite weighted sum of the pixel values I_n ($n \in \Theta_N$):

$$\hat{\mathscr{I}}(\omega) = \sum_{n\in\Theta_N} I_n \eta(\omega_n - \omega) = \sum_{n\in\Theta_N} I_n \eta_n^{\omega}, \tag{3}$$

η being a continuous reconstruction kernel and η_n^{ω} being the discrete kernel induced by sampling η translated in ω on the sampling grid.

The same consistency between continuous and discrete domain will imply that $\forall \omega \in \Omega, \sum_{n\in\Theta_N} \eta(\omega - \omega_n) = \sum_{n\in\Theta_N} \eta_n^{\omega} = 1$: sampling a reconstruction kernel translated at any location $\omega \in \Omega$ leads to a discrete summative kernel. Moreover, if the sampling is uniform, then η is even.

3.2.2 Kernel-based image processing

Let φ be a linear spatial domain operation transforming a continuous image $\mathscr{I}$ into another continuous image $\mathscr{I}'$ such that $\forall \omega \in \Omega$, $\mathscr{I}'(\omega) = \mathscr{I}(\varphi(\omega))$. Let ψ be the inverse of φ, i.e. $\forall \omega \in \Omega$, $\varphi(\psi(\omega)) = \psi(\varphi(\omega)) = \omega$. Let $\{\omega_n\}_{n\in\Theta_N}$ be the sampling locations and $\{I_n\}_{n\in\Theta_N}$ be the pixel values of the original discrete image – supposedly obtained by sampling $\mathscr{I}$.

Kernel based image processing consists in deriving a discrete operation that is equivalent to sampling $\mathscr{I}'$ on the sampling grid.

Sampling the continuous image $\mathscr{I}'$ leads to the sampled image I' such that:

$$\forall k \in \Theta_N, I'_k = \int_{\mathbb{R}^2} \mathscr{I}'(\omega)\kappa(\omega - \omega_k)d\omega, \tag{4}$$

κ being the sampling kernel. Considering $\mathscr{I}'(\omega) = \mathscr{I}(\varphi(\omega))$, Equation 4 becomes:

$$\forall k \in \Theta_N, I'_k = \int_{\mathbb{R}^2} \mathscr{I}(\varphi(\omega))\kappa(\omega - \omega_k)d\omega. \tag{5}$$

Now, let η be a reconstruction kernel, i.e. $\forall \omega \in \Omega$, $\mathscr{I}(\omega) = \sum_{n\in\Theta_N} I_n \eta(\omega_n - \omega)$, then $\forall k \in \Theta_N$:

$$\begin{aligned} I'_k &= \int_{\mathbb{R}^2} \sum_{n\in\Theta_N} I_n \eta(\varphi(\omega_n - \omega))\kappa(\omega - \omega_k)d\omega \\ &= \sum_{n\in\Theta_N} I_n \int_{\mathbb{R}^2} \eta(\varphi(\omega_n - \omega))\kappa(\omega - \omega_k)d\omega = \sum_{n\in\Theta_N} I_n \rho_n^k, \end{aligned} \tag{6}$$

with $\rho_n^k = \int_{\mathbb{R}^2} \eta(\varphi(\omega_n - \omega))\kappa(\omega - \omega_k)d\omega$.
By two simple variable changes $(\omega_n - \omega) \to \omega$ then$\varphi(\omega) \to \omega_n$, ρ_n^k can be rewritten in:

$$\rho_n^k = \int_{\mathbb{R}^2} \eta(\omega_n - \omega)\kappa(\omega_n - \omega_k - \psi(\omega_n - \omega))\frac{\delta\psi}{\delta\omega}(\omega - \omega_k)d\omega. \tag{7}$$

Thus, estimating the discrete values of I' based on I comes down to defining, for each location k, a positive discrete kernel ρ^k by sampling the transformed kernel κ^ψ, defined by $\forall \omega \in \Omega$, $\kappa^\psi(\omega) = \kappa(\psi(\omega))\frac{\delta\psi}{\delta\omega}(\omega)$, on the sampling grid formed by translating the interpolation kernel η on each sampling location.

Let C_n be the (possibly non normalized) fuzzy subset whose membership function is defined by: $\forall \omega \in \Omega$, $\mu_{C_n}(\omega) = \eta(\omega - \omega_n)$. Since η is a positive even kernel, by construction $\forall \omega \in \Omega$, $\sum_{n\in\Theta_N} \mu_{C_n}(\omega) = 1$. Therefore the subsets $\{C_n\}$ $(n \in \Theta_N)$ form a fuzzy partition.

Now, let $Q^k_{\kappa^\psi}$ be the measure defined by: $\forall A \subseteq \Theta_N$, $Q^k_{\kappa^\psi}(A) = \sum_{n\in A} \rho_n^k$, the value I'_k can be seen as being the estimate of I w.r.t. $Q^k_{\kappa^\psi}$:

$$I'_k = \mathbb{E}_{Q^k_{\kappa^\psi}}(I). \tag{8}$$

The measure $Q^k_{\kappa^\psi}$ can also be seen as an estimate:

$$\begin{aligned} \forall A \subseteq \Theta_N, \quad Q^k_{\kappa^\psi}(A) &= \int_{\mathbb{R}^2} \mu_{C_n}\kappa(\psi(\omega - \omega_k))\frac{\delta\psi}{\delta\omega}(\omega)d\omega \\ &= \int_{\mathbb{R}^2} \kappa(\psi(\omega - \omega_k))\frac{\delta\psi}{\delta\omega}(\omega)\sum_{n\in A}\mu_{C_n}d\omega \\ &= \int_{\mathbb{R}^2} \kappa(\psi(\omega - \omega_k))\frac{\delta\psi}{\delta\omega}(\omega)\Upsilon_A(\omega)d\omega, \end{aligned} \tag{9}$$

Υ_A being defined in Definition 2. Thus Equation 9 becomes:

$$\forall A \subseteq \Theta_N, Q^k_{\kappa^\psi}(A) = \mathbb{E}_{P_{\kappa^\psi}}(\Upsilon_A), \tag{10}$$

P_{κ^ψ} being the set measure whose density is defined by the kernel κ^ψ.

As an example, let φ be a rigid transformation, then $\varphi(\omega) = R.\omega + T$, where R is a rotation matrix and T a translation vector. Then $\psi(\omega) = R^T.\omega - R^T.T$ $\bullet^T$ being the transpose operator and $\frac{\delta\psi}{\delta\omega}(\omega) = R^T$. Therefore, $\forall\omega \in \Omega,\ \kappa^\psi(\omega) = R^T.\kappa(R^T\omega - R^T.T)$. This is illustrated in Figure 6.

3.2.3 Maxitive kernel-based image processing

The maxitive extension of the kernel-based image processing simply consists in replacing the summative sampling kernel by a maxitive sampling kernel. This replacement aims at representing imprecise knowledge on the sampling kernel, either because the original sampling kernel is unknown, either because the kernel that would lead to the least distortion in the transformed image is unknown.

Let π be a maxitive kernel. Replacing κ by π in Equation 2 leads to an interval-valued discrete image $\underline{\overline{I}}$ such that $\forall n \in \Theta_N$, $\underline{\overline{I}}_n = \underline{\overline{\mathbb{E}}}_{\Pi_{\pi^{\omega_k}}}(\mathscr{I})$, $\Pi_{\pi^{\omega_k}}$ being the possibility measure induced by π^{ω_k}, the maxitive kernel π translated on ω_k. This interval valued discrete image represents the convex set of all the discrete images that could have been obtained by sampling $\mathscr{I}$ with a kernel $\kappa \in \mathscr{M}(\pi)$ (see [14]).

Let π^ψ be the maxitive kernel π transformed by the application ψ such that: $\forall\omega \in \Omega$, $\pi^\psi(\omega) = \pi(\psi(\omega))\frac{\delta\psi}{\delta\omega}(\omega)$. Then, Expression 10 can be extended to define the capacity ν^k, for each location ω_k ($k \in \Theta_N$):

$$\forall A \subseteq \Theta_N, \nu^k_{\pi^\psi}(A) = \overline{\mathbb{E}}_{\Pi_{\pi^\psi}}(\Upsilon_A). \tag{11}$$

By construction, $\nu^k_{\pi^\psi}$ is a concave capacity such that $\forall\kappa \in \mathscr{M}(\pi)$ and $\forall A \subseteq \Theta_N$, $Q^k_{\kappa^\psi}(A) \leq \nu^k_{\pi^\psi}(A)$.

The final step of this extension consists of estimating the interval valued image $\underline{\overline{I}}'$ by replacing, in Equation 8, the discrete additive set measure $Q^k_{\kappa^\psi}$ by the discrete non additive set measure $\nu^k_{\pi^\psi}$. This leads to:

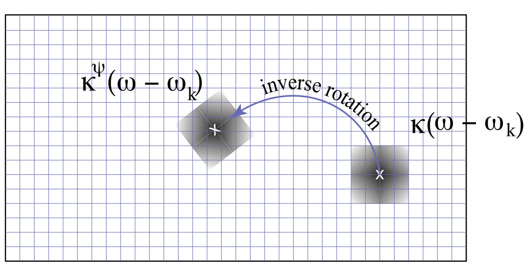

Fig. 6 Transformation of the sampling kernel κ translated in ω_k when φ is a rigid transformation.

$$\forall k \in \Theta_N, \overline{\underline{I_k}}' = \overline{\underline{\mathbb{E}}}_{\nu^k_{\pi^\Psi}}(I). \tag{12}$$

The estimation operator $\overline{\underline{\mathbb{E}}}$ propagates imprecise knowledge of the sampling kernel to the interval-valued transformed image.

3.3 Example: rigid transformations

Computation of the transformed maxitive kernel π^Ψ is as easy as computing the transformed summative kernel κ^Ψ. But one instrumental question is "which maxitive kernel has to be chosen?". As in image processing, mostly separable kernels or radial kernels are considered, a very interesting kernel would be a one that dominates any sampling kernel whose support is bounded. This problem has been addressed in [6] by considering separable kernels and separable estimations. As mentioned by the authors, such an approach is not suitable for transformations like rotations. A nice answer to this problem is proposed in [1] that gives different $2D$ maxitive kernels depending on which summative kernels have to be considered. For example, the continuous maxitive kernel $\breve{\pi}(\omega) = \max(0, 1 - \|\omega\|^2)$ dominates every bell-shaped radial continuous summative kernel whose support is included in $[-1, 1]$. This is very convenient because it can represent the fact that the support of the sampling kernel is at most the distance between two pixels but all what is known about the shape is that it is bell-shaped and radial. Note that the Dirac impulse is included in $\mathscr{M}(\breve{\pi})$, ensuring a kind of guaranteed preservation of the information carried on by the digital original image.

Thus now let us choose $\breve{\pi}$ to be the sampling maxitive kernel, then $\forall \omega \in \Omega$, $\breve{\pi}^\Psi(\omega) = \breve{\pi}(R^T \omega - R^T.T)$. For seek of simplicity, let us choose the partition $\{C_n\}$ $(n \in \Theta_N)$ to be a crisp partition, each C_n being a box centered on ω_n. Then Equation 11 leads to:

$$\forall A \subseteq \Theta_N, \nu^k_{\pi^\Psi}(A) = \overline{\mathbb{E}}_{\Pi_{\pi^\Psi}}(\Upsilon_A) = \sup_{\omega \in \bigcup_{n \in A} C_n} \breve{\pi}(R^T \omega - R^T.T).$$

In that case $\nu^k_{\pi^\Psi}$ is a possibility measure associated to the discrete possibility distribution β^k defined by:

$$\forall n \in \Theta_N, \beta^k_n = \sup_{\omega \in C_n} \breve{\pi}(R^T \omega - R^T.T).$$

If the partition is not crisp, then $\nu^k_{\pi^\Psi}$ is not a possibility measure but can be computed analytically (see [7] for an example of such a computation).

4 Conclusion and discussion

In many applications, images are subject to geometric distortions introduced by perspective, misalignment, optical aberrations, movement of the imaging sensor etc. which need to be corrected for further interpretation. Digital geometrical transformations are instrumental for reversing these distortions, align different images in a common frame or simply enable image contain analyse or interpretation. Many of those transformations aim a mimicking a physical operation defined in the continuous illumination domain. Kernel-based approach is a convenient way for designing digital transformations that may ensure a certain preservation of digital image topological and illumination properties. However, this method relies on modeling the interplay between continuous and discrete domain via two arbitrarily chosen kernels that model the sampling (to go from continuous to discrete domain) and the reconstruction (to go from discrete to continuous domain) operations. The arbitrariness of this choice can have severe consequences in applications where details have to be preserved that are comparable with image resolution.

Maxitive image processing can be an interesting solution to preserve the information carried on by the original image. It allows to compute de (convex) set of all images that would have been obtained by considering a (convex) set of kernels that could have been appropriate to compute this transformation. This computation can be achieved with a very low increase of the computation complexity (see e.g. [6]).

This article is focussed on modeling imprecise knowledge on the sampling kernel. Imprecise knowledge on the approximation kernel has to be taken into account in another way. Note however that, due to the interchangeable role of kernels in Expression 7, this modeling in fact addresses imprecise knowledge on the convolution of both reconstruction and sampling kernels (see e.g. [8]). This needs to be further investigated. Other problems have to be investigated including reversibility – how both illumination and topological information preserved when an image is subject to a transformation and then its inverse transformation? – and selection, in the convex set of obtained images, of a representative image to be presented to the expert (see e.g. [11]).

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