On counting orientations for graph homomorphisms and for dually embedded graphs using the Tutte polynomial of matroid perspectives

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Abstract The goal of this note is to highlight applications to graph orientations of some recent results in oriented matroid perspectives, by providing adapted settings and a detailed example. An (oriented) matroid perspective (or morphism, or strong map, or quotient) is an ordered pair of (oriented) matroids satisfying some structural relationship. In the case of graphs, two notable types of perspectives can be considered: graph homomorphisms, and dually embedded graphs on a surface. The Tutte polynomial of such a perspective is a classical polynomial (also called Las Vergnas polynomial in the case of dually embedded graphs), whose coefficients and (some) evaluations are known to count pairs of orientations of certain types. We show how coefficients and (other) evaluations of the polynomial also count pairs of orientations of certain types where some edge orientations are fixed, as well as some equivalence classes of pairs of orientations of certain types. These properties appear when the edge set is linearly ordered.

1 Perspectives for graphs

This paper briefly highlights applications to graph orientations of some results recently published in [6], by providing adapted settings and a detailed example. In general, an (oriented) matroid perspective (or morphism, or strong map, or quotient, up to unimportant variants) is an ordered pair $M \to N$ of (oriented) matroids on the same ground set satisfying some structural relationship, that can be characterized in various ways, such as: any circuit of M is a (conformal) union of circuits of N, or, equivalently, any cocircuit of N is a (conformal) union of cocircuits of M (see [13] or [18, Section 7.3] on matroid perspectives, and [2, Section 7.7] on oriented matroid perspectives). In the case of graphs, two notable types of perspectives can be considered, coming from graph homomorphisms and from graph embeddings.

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When we consider cycles and cuts of a graph, we always consider them as subsets of edges (vertices are not relevant here). We denote by $M_c(G)$ the cycle matroid of the graph G (whose circuits and cocircuits are inclusion-minimal cycles and cuts of G, respectively).

First situation - Graph homomorphism. Consider an homomorphism between two graphs G and H, in the usual sense where vertices of G are merged to get vertices of H, while keeping a natural bijection between edges of G and H. Then any cycle of G corresponds to a union of cycles of H, and any cut of H corresponds to a union of cuts of G. (In what follows, we write "is" rather than "corresponds to".) Then the two cycle matroids of these graphs form a perspective $M_c(G) \to M_c(H)$. One can focus on the following usual general construction: given a graph $\overline{G} = (V, \overline{E})$ and $A \subseteq \overline{E}$, the two graphs $G = \overline{G} \backslash A$ and $H = \overline{G}/A$ form such an homomorphism. An example is given in Figure 1.

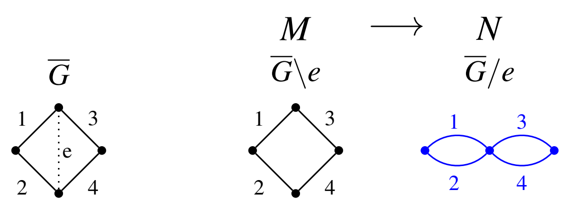


Fig. 1: Perspective $M_c(G) \to M_c(H)$ from a graph homomorphism between G and H, with $G = \overline{G}\backslash e$ and $H = \overline{G}/e$ (the role of the edge e of $\overline{G}$ is to identify which vertices are merged to get H from G).

Second situation - Dually embedded graphs. Consider a graph H embedded on a closed orientable surface, and a graph H^* dually embedded on the same surface, in the following sense. Vertices of H^* correspond to regions of the embedding of H (connected components of the surface after deleting the embedding of H). Edges of H^* bijectively correspond to edges of H, in such a way that an edge of H^* joins the two regions separated by (the embedding of) its corresponding edge of H. Then the two matroids $M_c(H^*)^*$ (shortly, the dual matroid of a dual embedding of H) and $M_c(H)$ form a perspective $M_c(H^*)^* \to M_c(H)$. As mentioned in [14], this is a consequence of a classical result from [3] (more precisely, this paper characterizes when two graphs can be dually embedded on a surface; it implies that the set of edges adjacent to any vertex of H^* corresponds to an eulerian subgraph of H, which implies that the above definition of a matroid perspective is satisfied, as cuts of H^* thus correspond to unions of cycles of H). An example is given in Figure 2.

Observe that if H is planar and embedded in the sphere, then $M_c(H^*)^* = M_c(H)$ (as matroid duality is consistent with planar graph duality) and then the perspective is trivial (that is, $M = N$). Let us mention that if H is cellularly embedded (that is, each region of the embedding is homeomorphic to an open disk whose boundary is a cycle of the graph), then the graph H^* and its embedding are uniquely determined (and cellularly embedded too). However, this assumption is not required here. One can see [4, Sections 2.1 and 4.1] for more details on this nuance and how to handle it in terms of polynomials. The example of Figure 2 is not cellularly embedded.

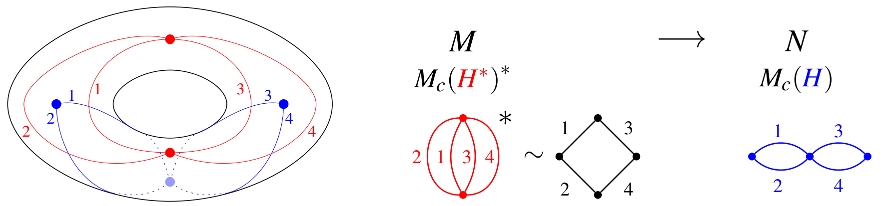


Fig. 2: Perspective $M_c(H^*)^* \to M_c(H)$ from an embedding of H (blue) and H^* (red) in the torus. Observe that, in this example, $M_c(H^*)^*$ yields the same matroid as that of the graph G from Figure 1. Hence, this example illustrates both situations (but the two situations are not equivalent in general, as H^* might not be a planar graph and thus $M_c(H^*)^*$ might not be a graphic matroid).

In each situation, considering an orientation of a graph in the pair directly yields a consistent orientation for the other graph. Precisely, for homomorphisms, use a consistent orientation of $\overline{G}$. And, for embeddings, use a $\uparrow\!\rightarrow$ rule, that is, if one directs an edge of H from left to right, then the corresponding edge in H^* is directed from bottom to top (that is, obtained by a quarter rotation counterclockwise). Then, the resulting pair of orientations effectively forms an oriented matroid perspective. We will consider the set of orientations of a perspective as the set 2^E, provided that a reference orientation, say $M \to N$, is given and then all orientations are obtained by reorienting some $A \subseteq E$ from this reference orientation, yielding $-_AM \to -_AN$.

2 Classical properties of the Tutte polynomial

The Tutte polynomial of a matroid perspective $M \to N$ is a 3-variable polynomial defined in [14, 16] in terms of rank functions. Ever since, this polynomial has been mainly studied by Michel Las Vergnas in a series of papers, including [15, 17]. See [8] for a recent survey. In the particular case of a trivial perspective where $M = N$, one retrieves the usual and well-known Tutte polynomial of the matroid M. In the above second situation for graphs, this polynomial was specifically studied and named Las Vergnas polynomial in [1], and was then surveyed and deepened in [4]. When the third variable is set to 1 (the third variable is not relevant in terms of orientations), it is given by: $T(M,N;x,y,1) \;=\; \sum_{A\subseteq E} (x-1)^{r(N)-r_N(A)}\;(y-1)^{|A|-r_M(A)}$.

Concerning orientations, the main theorem of [15] states that, for an oriented matroid perspective $M \to N$ on a linearly ordered set E, its Tutte polynomial satisfies

$$T(M,N;x,y,1) = \sum_{A\subseteq E} \left(\frac{x}{2}\right)^{|O^*(-_AN)|} \left(\frac{y}{2}\right)^{|O(-_AM)|}, \text{ where}$$

$$
\begin{aligned}
O(M) &= \{\ \min(C) \ \mid\ C \text{ positive circuit of } M\ \} && \text{(for a general oriented matroid } M\text{)}\\
&= \{\ \min(C) \ \mid\ C \text{ directed cycle of } G\ \} && \text{(when } M = M_c(G)\text{, 1st situation)}\\
&= \{\ \min(C) \ \mid\ C \text{ directed cut of } H^*\ \} && \text{(when } M = M_c(H^*)^*\text{, 2nd situation),}\\
O^*(N) &= \{\ \min(C) \ \mid\ C \text{ positive cocircuit of } N\ \} && \text{(for a general oriented matroid } N\text{)}\\
&= \{\ \min(C) \ \mid\ C \text{ directed cut of } H\ \} && \text{(when } N = M_c(H)\text{, both situations).}
\end{aligned}
$$

The sets $O(M)$ and $O^*(N)$ are known as the sets of *active* and *dual-active* elements of M and N, respectively. Note that they are two dual notions, as $O(M^*) = O^*(M)$. Observe that $O(M) = \emptyset$ if and only if M is acyclic, and that $O^*(N) = \emptyset$ if and only if N is totally cyclic (that is, strongly connected for a connected digraph). Let us denote by $T_{i,j}$ the coefficient of $x^i y^j$ of T. This number is an invariant (it does not depend on the ordering), and the above theorem can be summed up as:

$$T_{i,j} = 1/2^{i+j} \times \#\{\text{orientations with } i \text{ dual-active elements and } j \text{ active elements}\}.$$

See [9] for a survey and geometric interpretations. Notably, we get the counting results of Table 1.

Table 1: Classical counting results.

Orientations $-_AM \to -_AN$ with	**are counted by**
no condition	$T(M,N;2,2,1)$
acyclic $-_AM$	$T(M,N;2,0,1)$
totally cyclic $-_AN$	$T(M,N;0,2,1)$
acyclic $-_AM$ and totally cyclic $-_AN$	$T(M,N;0,0,1)$

Naturally, the useful translation of the terms “acyclic” and “totally cyclic” depends on the situation considered for graphs. For instance, for connected graphs, $T(M,N;0,0,1) = T_{0,0}$ counts, in the first situation, the number of orientations such that G is acyclic and H is strongly connected, and, in the second situation, the number of orientations such that both H and H^* are strongly connected.

3 More involved notions and results of the same flavour

The results below from [6] are obtained by means of active partitions which refine active elements, and by means of a partition of the set of orientations into activity classes. These notions for digraphs and oriented matroids were defined and used in the context of the active bijection in a series of papers, including [5, 10, 11, 12] (the Tutte polynomial also counts objects related to spanning trees or bases, see [9] for a survey). This corresponds to trivial perspectives (i.e., $M = N$). These notions are generalized in [6] for perspectives, where one can also find a 4-variable expansion formula for the Tutte polynomial, which can be seen as an algebraic counterpart of these constructions, and which was announced in [17]. The interpretations in the two situations for graphs follow the terminology presented in the previous section.

Definition 1. Let $M \to N$ be a perspective on a linearly ordered set E and $A \subseteq E$. The reorientation $-_AM \to -_AN$ is called *active-fixed* if no active element of $-_AM$

belongs to A, and *dual-active-fixed* if no dual-active element of $-_AN$ belongs to A. In other words, (dual-)active-fixed means that every (dual-)active element has the same orientation as in the reference orientation.

Definition 2. Let $M \to N$ be a perspective on a linearly ordered set E with $O(M) = \{g_1, \ldots, g_j\}_<$ and $O^*(N) = \{h_1, \ldots, h_i\}_<$. The *active filtration* of $M \to N$ is sequence of sets

$$\emptyset = G_j \subset \ldots \subset G_1 \subset G_0 \subseteq H_0 \subset H_1 \subset \ldots \subset H_i = E$$

defined by:

- $G_j = \emptyset$ and $E \setminus H_i = \emptyset$;
- for $0 \leq k \leq j-1$, G_k is the union of all positive circuits of M whose smallest element is greater than or equal to g_{k+1} (use directed cycles of G in the first graph situation, or directed cuts of H^* in the second graph situation);
- and, for $0 \leq k \leq i-1$, $E \setminus H_k$ is the union of all positive cocircuits of N whose smallest element is greater than or equal to h_{k+1} (use directed cuts of H in the two graph situations).

Observe that $H_0 \setminus G_0$ is the (possibly empty) set of elements belonging to positive cocircuits of M and positive circuits of N. (If $M = N$, then $H_0 \setminus G_0 = \emptyset$; and if M is acyclic and N totally cyclic then $H_0 \setminus G_0 = E$.)

The successive differences of these sets form a partition of E, named the *active partition* of $M \to N$, with *j cyclic parts* contained in G_0, *i acyclic parts* contained in $E \setminus H_0$, and a (possibly empty) *hybrid part* $H_0 \setminus G_0$. The smallest elements of the (a)cyclic parts are the (dual-)active elements.

Then, the *activity class of $M \to N$* is the set of 2^{i+j} reorientations obtained by arbitrarily reorienting cyclic and/or acyclic parts of the active partition of $M \to N$ (or, equivalently, subsets G_k, $0 \leq k \leq j-1$, and/or $E \setminus H_k$, $0 \leq k \leq i-1$, given by its active filtration). Observe that the hybrid part is not reoriented.

Theorem 1 *Let $M \to N$ be a perspective on a linearly ordered set E. Given a reorientation of $M \to N$, all reorientations in the same activity class as this reorientation share the same active filtration/partition. Activity classes of orientations form a partition of the set 2^E of orientations (into boolean lattices, actually). In each activity class of orientations, there is one and only one orientation which is active-fixed and dual-active-fixed. We also get the counting results below and that of Table 2.*

$$\begin{aligned} T_{i,j} &= \#\{ \text{ active-fixed dual-active-fixed orientations} \\ &\qquad\qquad \text{with } i \text{ dual-active elements and } j \text{ active elements } \} \\ &= \#\{ \text{ activity classes of orientations} \\ &\qquad\qquad \text{with } i \text{ dual-active elements and } j \text{ active elements } \} \end{aligned}$$

Table 2: Counting results of Theorem 1, completing Table 1.

Orientations $-_A M \to -_A N$ with	**Activity classes with**	**are counted by**
acyclic $-_A M$ and dual-active-fixed $-_A N$	acyclic $-_A M$	$T(M,N;1,0,1)$
active-fixed $-_A M$ and totally cyclic $-_A N$	totally cyclic $-_A N$	$T(M,N;0,1,1)$
active-fixed $-_A M$		$T(M,N;2,1,1)$
dual-active-fixed $-_A N$		$T(M,N;1,2,1)$
active-fixed $-_A M$ and dual-active-fixed $-_A N$	no condition	$T(M,N;1,1,1)$
	acyclic $-_A M$ and totally cyclic $-_A N$	$T(M,N;0,0,1)$

4 A detailed example

Let us illustrate Table 1, Theorem 1, and Table 2 for the two graph situations on the example of Figures 1 and 2. See Figure 3 and details below. We have

$$T(M,N;x,y,1) = 4\left(\frac{x}{2}\right)^2 + 8\left(\frac{x}{2}\right) + 2\left(\frac{y}{2}\right) + 2 = x^2 + 4x + y + 2,$$

$$\begin{array}{lll} T(M,N;2,2,1) = 16, & T(M,N;1,1,1) = 8, & T(M,N;0,0,1) = 2, \\ T(M,N;2,1,1) = 15, & T(M,N;2,0,1) = 14, & T(M,N;1,0,1) = 7, \\ T(M,N;1,2,1) = 9, & T(M,N;0,2,1) = 4, & T(M,N;0,1,1) = 3. \end{array}$$

Figure 3 shows the list of orientations for the perspective of Figures 1 and 2. We use the natural ordering $1 < 2 < 3 < 4$. Only half the orientations are represented (the other half are the opposites, with the same active elements, dual-active elements, and active partitions). The reference orientation is the first of the list (with $A = \emptyset$). At each orientation, we indicate the set A of reoriented edges, the active and dual-active elements, and the active partition below. Next, we indicate (with a $\times$ symbol) if it is active-fixed, dual-active-fixed, and the same information below for its opposite.

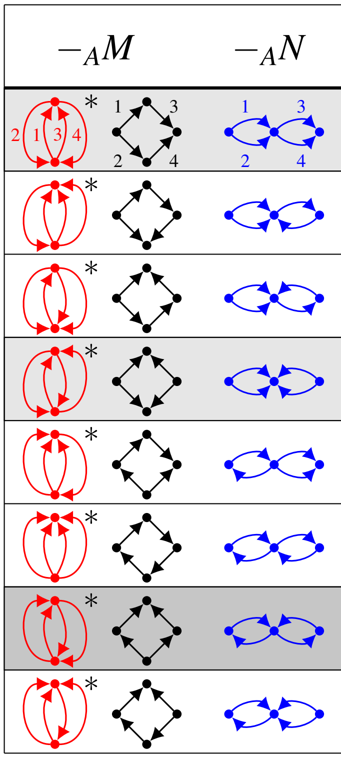
Let us detail the $T(M,N;1,1,1) = 8$ activity classes.

For the two light grey cells at the first and fourth rows: these orientations and their opposites are all in the same activity class of size 4 given by $A \in \{\emptyset, 34, 12, 1234\}$.

For the dark grey cell at the seventh row: this orientation and its opposite are those whose hybrid part is equal to E, and each forms an activity class of size 1 (one for $A = 23$ and the other for $A = E \setminus 23 = 14$). These special classes are also enumerated by $T(M,N;0,0,1) = 2$ (as having an acyclic $-_A M$ and a totally cyclic $-_A N$).

Every other activity class has size 2, it consists of:

- one orientation represented in a cell and its opposite (when the hybrid part is empty, that is, for $A \in \{24, 13\}$),
- or of one orientation and the opposite of an orientation represented in another cell (there are two classes of this kind, for $A \in \{4, 124\} = \{4, E \setminus 3\}$, and for $A \in \{3, 123\} = \{3, E \setminus 4\}$),



$O(-_A M)$ cyclic parts	$O^*(-_A N)$ acyclic parts	$-_A$ hybrid part	active-fixed opposite	dual-active-fixed- opposite
∅ ∅	**13** 12+34	$-_\emptyset$ ∅	× ×	× ·
∅ ∅	**1** 12	$-_4$ 34	× ×	× ·
∅ ∅	**1** 12	$-_3$ 34	× ×	× ·
∅ ∅	**13** 12+34	$-_{34}$ ∅	× ×	· ·
∅ ∅	**3** 34	$-_2$ 12	× ×	× ·
1 1234	**∅** ∅	$-_{24}$ ∅	× ·	× ×
∅ ∅	**∅** ∅	$-_{23}$ 1234	× ×	× ×
∅ ∅	**3** 34	$-_{234}$ 12	× ×	· ×

Fig. 3: Detailing all orientations for the example of Figures 1 and 2. See Section 4.

- or of two represented orientations (for $A \in \{2, 234\}$),
- or of the opposite orientations of the latter two (for $A \in \{134, 1\}$).

There are $T(M,N;1,1,1) = 8$ orientations which are both active-fixed and dual-active-fixed. They have a $\times$ symbol for each of these two boxes, and they are given by $A \in \{\emptyset, 4, 3, 2, 24, 23, E \setminus 23 = 14, E \setminus 234 = 1\}$. In each activity class above, there is exactly one such orientation which is both active-fixed and dual-active-fixed.

Amongst the activity classes, one can count only the $T(M,N;1,0,1) = 7$ ones with acyclic $-_A M$. They are those which have an empty cyclic part amongst the classes conisdered above, that is, all the 8 classes but the class given by $A \in \{24, 13\}$. Equivalently, $T(M,N;1,0,1) = 7$ counts the number of orientations with acyclic $-_A M$ and dual-active-fixed $-_A N$ (they have an empty cyclic part and a $\times$ symbol in the dual-active-fixed box). One can also count the $T(M,N;1,2,1) = 9$ orientations with dual-active-fixed $-_A N$ (which have a $\times$ symbol in the dual-active-fixed box).

Similarly, one can count the $T(M,N;0,1,1) = 3$ activity classes with totally cyclic $-_A N$. They have an empty acyclic part, and they are given by the two classes of the dark grey cell at the seventh row (with one orientation each), plus the class at the sixth row (formed by two opposite orientations). One can count the $T(M,N;0,2,1) = 4$ orientations with totally cyclic $-_A N$, which are those having an empty acylic part, or equivalently those belonging to the 3 previous classes (they are given by $A \in \{24, E\setminus 24 = 13, 23, E\setminus 23 = 14\}$).

The coefficient $T_{1,0} = 4$ counts dual-active-fixed orientations (or activity classes) with 1 dual-active element and 0 active element. They are given by $A \in \{4,3,2,E\setminus 234 = 1\}$ (or their classes as described above).

Other coefficients and evaluations can be looked at in similar ways.

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