Some Problems in Graph Coloring: Methods, Extensions and Results

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Université Montpellier 2

Laboratoire d'Informatique, de Robotique et de Microélectronique

Habilitation à diriger des recherches

École Doctorale Information, Structures, Systèmes

Alexandre Pinlou

Some Problems in Graph Coloring: Methods, Extensions and Results

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Abstract

The « Habilitation à Diriger des Recherches » is the occasion to look back on my research work since the end of my PhD thesis in 2006. I will not present all my results in this manuscript but a selection of them: this will be an overview of eleven papers which have been published in international journals or are submitted and which are included in annexes. These papers have been done with different coauthors: Marthe Bonamy, Daniel Gonçalves, Benjamin Lévêque, Amanda Montejano, Mickaël Montassier, Pascal Ochem, André Raspaud, Sagnik Sen and Éric Sopena. I would like to thanks them without whom this work would never have been possible.

I also take this opportunity to thank all my other co-authors: Luigi Addario-Berry, François Dross, Louis Esperet, Frédéric Havet, Ross Kang, Daniel Král', Colin McDiarmid, Michaël Rao, Jean-Sébastien Sereni and Stéphan Thomassé. Working with you is always a pleasure !

Since the beginning of my PhD, I have been interested in various fields of graph theory, but the main topic that I work on is the graph coloring. In particular, I have studied problems such as the oriented coloring, the acyclic coloring, the signed coloring, the square coloring, ... It is then natural that this manuscript gathers results on graph coloring. It is divided into three chapters. Each chapter is dedicated to a method of proof that I have been led to use for my research works and that has given results described in this manuscript. We will present each method, some extensions and the related results. The lemmas, theorems, and others which I took part are shaded in this manuscript.

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The entropy compression method

In the first chapter, we present a recent tool dubbed the *entropy compression method* which is based on the Lovász Local Lemma. The Lovász Local Lemma was introduced in the 70's to prove results on 3-chromatic hypergraphs [EL75]. It is a remarkably powerful probabilistic method to prove the existence of combinatorial objects satisfying a set of constraints expressed as a set of bad events which must not occur. However, one of the weakness of the Lovász Local Lemma is that it does not indicate how to efficiently avoid the bad events in practice.

A recent breakthrough by Moser and Tardos [MT10] provides algorithmic version of the Lovász Local Lemma in quite general circumstances. To do so, they used a new species of monotonicity argument dubbed the *entropy compression method*. This Moser and Tardos' result was really inspiring and Grytczuk, Kozik and Micek [GKM13] adapted the technique for a problem on combinatorics on words. This nice adaptation seems to be applicable to coloring problems, but not only, whenever the Lovász Local Lemma is, with the benefits of providing better bounds. For example, the entropy compression method has been used to get bounds on non-repetitive coloring [DJKW14] that improve previous results using the Lovász Local Lemma and on acyclic-edge coloring [EP13].

In this context, we developed a general framework that can be applied to most of coloring problems. We then applied this framework and we get the best known bounds, up to now, for the acyclic chromatic number of graphs with bounded degree, non-repetitive chromatic number of graphs with bounded degree, facial Thue chromatic index of planar graphs, ... We also applied the entropy compression method to problems on combinatorics on words: we recently solved an old conjecture on pattern avoidance.

This chapter is based on joint works with Daniel Gonçalves, Mickaël Montassier and Pascal Ochem.

[GMP14] D. Gonçalves, M. Montassier, and A. Pinlou. Entropy compression method applied to graph colorings. *Submitted*, 2014. Available on Arxiv:1406.4380.

[OP14a] P. Ochem and A. Pinlou. Application of entropy compression in pattern avoidance. *The Electronic Journal of Combinatorics*, 21(2):1–12, 2014.

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Graph homomorphisms and graph colorings

In this chapter, we present some notions of graph colorings from the point of view of graph homomorphisms. It is well-known that a proper k-coloring of a simple graph G corresponds to a homomorphism of G to K_k. Considering homomorphisms from a more general context, we get a natural extension of the classical notion of coloring. We present in this chapter the notion of homomorphism of (n, m)-colored mixed graphs (graphs with arcs of n different types and edges of m different types) and the related notions of coloring. This has been introduced by Nešetřil and Raspaud [NR00] in 2000 as a generalization of the classical notion of homomorphism. We then present two special cases, namely homomorphisms of $(1, 0)$-colored mixed graphs (which are known as oriented homomorphisms) and homomorphisms of $(0, 2)$-colored mixed graphs (which are known as signed homomorphisms).

While dealing with homomorphisms of graphs, one of the important tools is the notion of universal graphs: given a graph family $\mathcal{F}$, a graph H is $\mathcal{F}$-universal if each member of $\mathcal{F}$ admits a homomorphism to H. When H is $\mathcal{F}$-universal, then the chromatic number of any member of $\mathcal{F}$ is upper-bounded by the number of vertices of H. We study some well-known families of universal graphs and we list their structural properties. Using these properties, we give some results on graph families such as bounded degree graphs, forests, partial k-trees, maximum average degree bounded graphs, planar graphs (with given girth), outerplanar graphs (with given girth), ...

Among others, we will present the Tromp construction which defines well known families of oriented and signed universal graphs. One of our major contributions is to study the properties of Tromp graphs and use them to get upper bounds for the oriented chromatic number and the signed chromatic number. In particular, up to now, we get the best upper bounds for the oriented chromatic number of planar graphs with girth 4 and 5: we get these bounds by showing that every graph of these two families admits an oriented homomorphism to some Tromp graph. We also get tight bounds for the signed chromatic number of several graph families, among which the family of partial 3-trees which admits a signed homomorphism to some Tromp graph.

This chapter is based on joint works with Amanda Montejano, Pascal Ochem, André Raspaud, Sagnik Sen and Éric Sopena.

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Coloring the square of graphs with bounded maximum average degree using the discharging method

The discharging method was introduced in the early 20th century, and is essentially known for being used by Appel, Haken and Kock [AH77, AHK77] in 1977 in order to prove the Four-Color-Theorem. More precisely, this technique is usually used to prove statements in structural graph theory, and it is commonly applied in the context of planar graphs and graphs with bounded maximum average degree.

The principle is the following. Suppose that, given a set S of configurations, we want to prove that a graph G necessarily contains one of the configuration of S. We assign a charge ω to some elements of G. Using global information on the structure of G, we are able to compute the total sum of the charges $\omega(G)$. Then, assuming G does not contain any configuration from S, the discharging method redistributes the charges following some discharging rules (the discharging process ensures that no charge is lost and no charge is created). After the discharging process, we are able to compute the total sum of the new charges $\omega^*(G)$. We then get a contradiction by showing that $\omega(G) \neq \omega^*(G)$.

Initially, the discharging method was used as a *local* discharging method. This means that the discharging rules was designed so that an element redistributes its charge in its neighborhood. However, in certain cases, the whole graph contains enough charge but this charge can be arbitrarily far away from the elements that are negative. In the last decade, the *global* discharging method has been designed. This notion of global discharging was introduced by Borodin, Ivanova and Kostochka [BIK07]. A discharging method is global when we consider arbitrarily large structures and make some charges travel arbitrarily far along those structures. In some sense, these techniques of global discharging can be viewed as the start of the "second generation" of the discharging method, expanding its use to more difficult problems.

The aim of this chapter is to present this method, in particular some progresses from the last decade, i.e. global discharging. To illustrate these progresses, we will consider the coloring of the square of graphs with bounded maximum average degree for which we obtained new results using the global discharging method. Coloring the square of a graph G consists to color its vertices so that two vertices at distance at most 2 get distinct colors (i.e. two adjacent vertices get distinct colors and two vertices sharing a common neighbor get distinct colors). This clearly corresponds to a proper coloring of the square of G. This coloring is called a *2-distance coloring*. It is clear that we need at least $\Delta + 1$ colors for any 2-distance coloring since a vertex of degree Δ together with its Δ neighbors form a set of $\Delta + 1$ vertices which must get distinct colors. We investigate this coloring notion for graphs with bounded maximum average degree and we characterize two thresholds. We prove that, for sufficiently large Δ, graphs with maximum degree Δ and maximum average degree less that $3 - \epsilon$ (for any $\epsilon > 0$) admit a 2-distance coloring with $\Delta + 1$ colors. For maximum average degree less that $4 - \epsilon$, we prove that $\Delta + C$ colors are enough (where C is a constant not depending on Δ). Finally, for maximum average degree at least 4, it is already known that $C'\Delta$ colors are enough. Therefore, thresholds of $3 - \epsilon$ and $4 - \epsilon$ are tight.

This chapter is based on joint works with Marthe Bonamy and Benjamin Lévêque.

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